ON THE CAPACITY OF MIMO BROADCAST CHANNELS WITH REDUCED FEEDBACK BY ANTENNA SELECTION

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ABSTRACT

In this paper we analyze the performance of random beamforming schemes in a multi-user Gaussian broadcast channel utilizing SINR feedback. For generality, the receivers are allowed to have multiple receive antennas. The first scheme analyzed allows each receiver to feed back the largest SINR it observes for each transmit beam. The distribution function of the maximum SINR is derived and is noted to differ from previous work. In an effort to further reduce feedback, a scheme where each user feeds back the maximum SINR observed over its receive antennas and transmit beams. Using the Fréchet bounds and properties of chi-squared random variables, the throughput of this system is bounded and empirically shown to approach the performance of the previous scheme as the number of users increases.

Index Terms:: multi-user MIMO, broadcast channel, random beamforming, LMMSE receivers

1. INTRODUCTION

The users in a broadcast channel experience varying levels of channel quality. For high throughput, it is useful for the transmitter to be fully aware of the channel to all the users. This represents a large amount of information that must be known at the transmitter. A mechanism by which the transmitter is aware of the channel state information is for each user to feed back the observed channel. The questions that concerns this paper are how to reduce the amount of feedback and the tradeoff between reduced feedback and performance. In this work, the random beamforming scheme suggested in [1] is considered. Because the throughput is a function of SINR, if each user sends back the SINR it experiences for each transmit beam, then the transmitter sends to the users that are currently experiencing the best channels for each beam. One possible scheme is to view each antenna in the system as an individual user and select the best antennsas. This scheme was utilized in [1].

This paper examines a scheme that further reduce the feedback information. A feedback scheme is proposed where each user sends back the maximum SINR experienced across all the receive antennas and across all the transmitted beams. Compared to viewing each antenna as an individual user, this scheme reduces the number of SINR values fed back to the transmitter per user.

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The paper is organized as follows. In Section 2, the system model and previous work is described. A new SINR distribution is derived. In Section 3, a reduced feedback scheme is analyzed where the receiver feeds back only the maximum observed SINR across receive antennas and transmit beams. Section 4 analyzes the asymptotic performance of the feedback schemes and Section 5 concludes the paper.

2. SYSTEM MODEL AND BACKGROUND

In this section the system model used to analyze the problem and previous work found in [1] are discussed. The results found in [1] will be built upon in the latter parts of this paper.

2.1. System Model

A block fading channel model is assumed for the Gaussian broadcast channel. The transmitter has M transmit antennas and there are n receivers, each with N receive antennas. It is further assumed that $n \gg M$ and $N \leq M$. Let $\mathbf{s}(t) \in \mathbb{C}^{M\times 1}$ be the transmitted vector of symbols at time slot t and $\mathbf{y}_i(t) \in \mathbb{C}^{N\times 1}$ be the received symbols at the i^{th} user at time slot t. The following model is used for the input-output relationship between the transmitter and the i^{th} user:

$$\mathbf{y}_i(t) = \sqrt{\rho_i} H_i \mathbf{s}(t) + \mathbf{w}_i(t), \quad i = 1, \ldots, n. \tag{1}$$

$H_i \in \mathbb{C}^{N\times M}$ is the complex channel matrix which is assumed to be known at the receiver, $\mathbf{w}_i \in \mathbb{C}^{N\times 1}$ is the white additive noise, and the elements of H_i and $\mathbf{w}_i$ are i.i.d. complex Gaussians with zero mean and unit variance as defined by Edelman ([5]). The transmit power is chosen to be M, i.e. $E\{\mathbf{s}^*\mathbf{s}\} = M$, the SNR at the receiver is $E\{\rho_i |H_i\mathbf{s}|^2\} = M\rho_i$ and ρ_i is the SNR of the i^{th} user. It is assumed that $\rho_i = \rho \, \forall i$.

2.2. Background

The random beamforming scheme developed in [1] forms the basis of this work. The key elements of this work are now described and expanded upon.

2.2.1. SINR Distribution

The transmission scheme, as developed in [1], involves generating M random orthonormal vectors $\phi_m \in \mathbb{R}^M$ for $m = 1, \ldots, M$, where ϕ_m are isotropically distributed in $\mathbb{R}^M$. Let $s_m(t)$ be the m^{th} transmit symbol at time t, then the total transmit signal at time slot t is given by

$$S(t) = \sum_{m=1}^{M} \phi_m(t) s_m(t). \tag{2}$$

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The received signal at the i^{th} user is given by

$$Y_i(t) = \sum_{m=1}^{M} H_i(t)\phi_m(t)s_m(t) + W_i(t). \quad (3)$$

Each receive antenna at each user is assumed to measure the SINR for each of the M transmitted beams and the maximum of the observed SINR values is fed back leading to the use of order statistics.

Assuming that the i^{th} user knows the quantity $H_i(t)\phi_m(t)$, from Equation (3) the SINR of the j^{th} receive antenna of the i^{th} user for the m^{th} transmit beam is computed by the following equation:

$$SINR_{i,j,m} = \frac{|H_{i,j}(t)\phi_m(t)|^2}{\frac{2}{\rho} + \sum_{k\neq m} |H_{i,j}(t)\phi_k(t)|^2}. \quad (4)$$

$H_{i,j}$ is the j^{th} row of the i^{th} user's channel matrix. It is well known that the numerator in Equation (4) is distributed as a $\chi^2(2)$ random variable and the denominator as a $\chi^2(2(M-1))$ random variable. The density of the SINR is given in [1]

$$f_s(x) = \left(\frac{cx}{2(1+x)^M} e^{-\frac{cx}{2}} + \frac{c+2(M-1)}{2(1+x)^M} e^{-\frac{cx}{2}} \right) u(x) \quad (5)$$

where, in our case, $c = \frac{2}{\rho}$. From now on, for notational simplicity, the $u(x)$ will dropped from the distribution and density expressions with the understanding that all the random variables of interest are nonnegative.

The distribution function, required for the order statistics, is given by the integration of the density in Equation (5) and yields Equation (6) at the top of the next page, where $\alpha = \frac{c}{2}$ and $Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$. The details of this calculation are omitted for brevity. This distribution function is different from that stated in [1]. To compare and verify these results, Figure 1 shows a plot of the distribution in [1], a plot of the distribution $F_s(x)$, and a plot of the numerical integration of the density in Equation (5). Because $F_s(x)$ differs from the work in [1], in order for the results derived in [1] to hold, the asymptotic properties that held for the distribution in [1] must be shown to hold for $F_s(x)$ derived above. Fortunately the properties hold and will be shown later in this paper.

2.2.2. *Scheduling Scheme*

Viewing each antenna as a separate user, the following approximation was shown in [1]:

$$\begin{aligned} R &\approx E\left\{ \sum_{m=1}^{M} \log\left(1 + \max_{i=1,\ldots,nN} SINR_{i,m}\right)\right\} \\ &= ME\left\{ \log\left(1 + \max_{i=1,\ldots,nN} SINR_{i,m}\right)\right\} \quad (7) \end{aligned}$$

The approximation sign is required because there is a small probability that the same antenna is the best for multiple transmit beams. To find the approximate rate requires the use of order statistics.

The distribution of the maximum SINR for a given beam is given by classical order statistics [2], [3] and is

$$F_{max}(x) = [F_s(x)]^{nN}, \quad (8)$$

where $F_s(x)$ is the CDF of the SINR. This equation can be used because the SINR values across antennas for a specific transmit beam

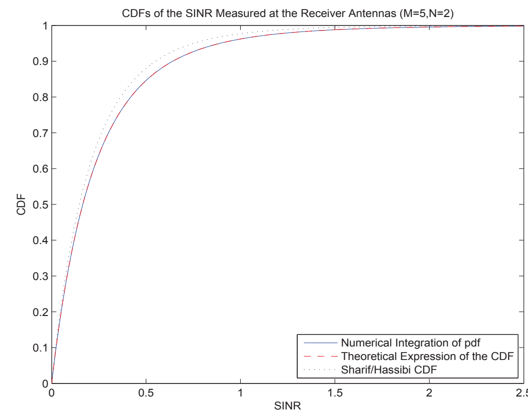

Fig. 1. SINR CDFs as measured at the receive antennas, SNR = 0 dB

are independent and marginally are identically distributed. Feeding back only the largest SINR value for each transmit beam yields nM total SINR values fedback in the system is considered in [1]. For future reference, the feedback scheme of [1] is referred to as **Scheme A**. The next question is how can the amount of feedback be reduced even further?

3. FEEDING BACK THE LARGEST SINR PER USER

Each user in **Scheme A** is feeding back the M SINR values associated with the largest SINR measured for each beam. To further reduce the amount of feedback, what happens if each user only feeds back the largest SINR value and the associated beam index? This would reduce the total amount of system feedback from nM SINR values to n SINR values and the corresponding n beam indices. This feedback scheme is referred to as **Scheme B**. The analysis of this scheme poses some difficulties that do not arise when considering **Scheme A**. This section addresses these new difficulties and derives bounds on the performance.

3.1. Analysis of Scheme B

Although **Scheme B** reduces the amount of feedback by a factor of M compared to **Scheme A**, it is suboptimal. This is due to the fact that the best SINR values for a particular beam may not be fed back due to the restriction that only one value can be sent back per user. This restriction, however, removes the mechanism which led to the approximation symbol in Equation (6). Although **Scheme B** may be suboptimal, in the interest of reducing the feedback as much as possible, this scheme is of interest.

The distribution of the SINR that is served by the transmitter is fundamentally different for **Scheme B** for two reasons. The first difference is that in **Scheme A**, the maximum is taken for each beam and the marginal distributions of the SINR are i.i.d. across receive antennas. However, the SINR values at a particular receive antenna for different transmit beams are coupled. To see this, fix the antenna at a particular user and vary the beam index. Let $|H_{i,j}(t)\phi_m(t)|^2 =$

$$F_s(x) = \left(1 - \alpha e^{\alpha}\left[\frac{(-1)^{M-1}\alpha^{M-2}Ei(-\alpha x)}{(M-2)!} + \frac{e^{-\alpha x}}{x^{M-2}}\sum_{k=0}^{M-3}\frac{(-1)^k\alpha^k x^k}{(M-2)(M-3)\cdots(M-2-k)}\right]\right.$$
$$\left.-e^{\alpha}(M-1)\left[\frac{(-1)^{M}\alpha^{M-1}Ei(-\alpha x)}{(M-1)!} + \frac{e^{-\alpha x}}{x^{M-1}}\sum_{k=0}^{M-2}\frac{(-1)^k\alpha^k x^k}{(M-1)(M-2)\cdots(M-1-k)}\right]\right) \quad (6)$$

X_m and $c = \frac{2}{\rho}$. Then the SINR values at the receive antenna are given by $\frac{X_1}{c+\sum_{k\neq 1} X_k}, \ldots, \frac{X_M}{c+\sum_{k\neq M}}$. The SINRs are coupled by the appearance of the numerator term of a particular SINR value appearing in the denominator of all the other SINR values. Because the SINR values are not independent, the order statistics used earlier cannot be applied. The second difference is that in **Scheme B** the number of SINR values to maximize over at the transmitter for each beam is a random quantity. These fundamental differences will now be addressed.

3.1.1. Bounding the Joint Distribution of the SINRs

Deriving the joint distribution of the SINRs for the transmit beams at a particular receive antenna is challenging. Rather than worrying about the explicit joint distribution, the distribution function is bounded and order statistics are applied to the bounds. The first bound of interest is due to Fréchet and is stated below.

Theorem 1. *([3], The Fréchet Bounds) Let $F(\mathbf{X})$ be an m-dimensional distribution function with marginals $F_j(x), 1 \leq j \leq m$. Then, for all $x_1, x_2, \ldots, x_m$*

$$\max\left(0, \sum_{j=1}^{m} F_j(x_j) - m + 1\right) \leq F(x_1, x_2, \ldots, x_m) \leq \min\left(F_1(x_1), \ldots, F_m(x_m)\right) \quad (9)$$

All the marginals are the same and are given by $F_s(x)$. So for the maximum order statistic, the Fréchet Bound simplifies to

$$\max\left(0, M\left(F_s(x) - 1\right) + 1\right) \leq F(x, \ldots, x) \leq F_s(x),$$

where $F(x, \ldots, x)$ is the joint distribution on $SINR_{i,j,1}, \ldots, SINR_{i,j,M}$ evaluated at x for each argument.

Upper bounding the distribution function by $F_s(x)$ is too loose. A tighter bound can be found in certain situations.

Theorem 2. *In the high and low SNR regime,*

$$\Pr\left(SINR_{i,j,1} \leq x, \ldots, SINR_{i,j,M} \leq x\right) \leq \left[F_s(x)\right]^M .$$

This also holds as the number of transmit antennas grows large for all SNR.

Proof. In the low SNR regime, the $2/\rho$ term in the denominator of Equation (4) dominates, so all the terms look like independent chi-squared random variables over the noise power. In this case, the joint trivially factorizes into the product of the marginals by independence. Note that

$$\Pr\left(\frac{1}{\frac{1}{(M-1)}}SINR \leq \frac{1}{\frac{1}{(M-1)}}x, \ldots, \frac{1}{\frac{1}{(M-1)}}SINR \leq \frac{1}{\frac{1}{(M-1)}}x\right)$$

is the same probability as above, and as M grows large, by the law of large numbers, the denominator in Equation (4) becomes constant, so the joint factors into the product of the marginals. For the high SNR regime, the summation term in Equation (4) dominates and by a change of variable, we can look at the equivalent joint distribution

$$\Pr\left(\frac{X_1}{S_n} \leq x', \ldots, \frac{X_n}{S_n} \leq x'\right)$$

where the X_i are the chi-squared numerators and $S_n = \sum_{i=1}^{n} X_i$. Then the $\frac{X_i}{S_n}$ are beta distributed and the joint distribution is Dirichlet, which is shown in [8],[9] to be less than the product of the beta distributed marginals. □

A general proof for all SNR has not been found, but it is conjectured from simulation results that it holds for all SNR. Applying this property and the lower Fréchet Bound yields

$$\left[\max\left(0, M\left(F_s(x) - 1\right) + 1\right)\right]^N \leq \left[F\left(SINR_{i,j,1} \leq x, \ldots, SINR_{i,j,M} \leq x\right)\right]^N \leq \left[F_s(x)\right]^{NM}. \quad (10)$$

Having bound the joint distribution, the maximum order statistic of a random number of observations is analyzed.

3.1.2. Order Statistics Over a Random Number of Observations

If each user feeds back the maximum observed SINR and the beam index that produced it, then the transmitter receives n SINR values and n beam indices over which to maximize. The number of SINR values to maximize over for a particular beam is, however, a random number. Because of the system model adopted, each beam has an equal probability of being the one that produced the maximum at any given user. The joint distribution on the number of SINR values fed back for each beam can then be viewed as a multinomial distribution where the probability of each beam being selected is $\frac{1}{M}$. The distribution of the order statistic of the maximum SINR is a distribution function raised to a power that is a random variable. Marginally, the selection of each beam is distributed binomially with probability $\frac{1}{M}$. Averaging over the binomially distributed exponent applied to the bounds in Equation (10) produces the bounds seen on the top of the following page.

3.1.3. Throughput Analysis

Using integration by parts and Equation (6), the throughput of a scheme is expressed as

$$R = ME\left\{\log\left(1 + X\right)\right\} = M\int_0^{\infty} \frac{1}{1+x}\left(1 - F(x)\right)dx. \quad (13)$$

where X is drawn from the distribution of the scheme being used. Numerical integration is very attractive since the closed form expression of the expectations in Equation (6) for the distributions derived earlier are not known to the authors. Figure (2) shows the throughput (or bounds on the throughput) as a function of the number of users. Notice that **Scheme A** and the bounds for **Scheme B** (and thus the

$$F_{\max_{lower}}(x) = \max\left\{0, \sum_{i=0}^{n} \left[M\left(F_s(x)-1\right)+1\right]^{Ni} \binom{n}{i} \left(\frac{1}{M}\right)^{i} \left(1-\frac{1}{M}\right)^{n-i}\right\} \quad (11)$$

$$F_{\max_{upper}}(x) = \sum_{i=0}^{n} \left[F_s(x)\right]^{NMi} \binom{n}{i} \left(\frac{1}{M}\right)^{i} \left(1-\frac{1}{M}\right)^{n-i}. \quad (12)$$

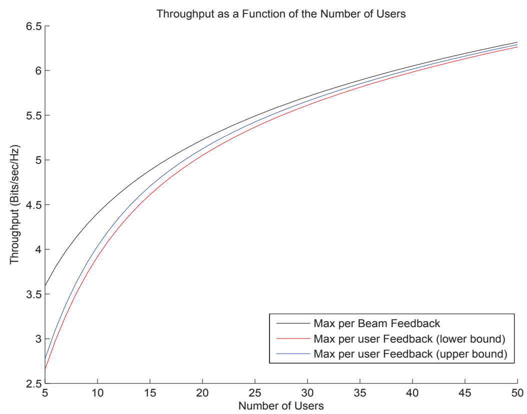

Fig. 2. Throughput as a Function of Number of Users for Different Schemes for M = 5, N = 2, SNR = 10 dB

actual distribution) tend towards each other. As the number of users in the system increases, it is expected that the maximum SINR for each beam is distributed across users with high probability. When the maximum SINR for each beam is distributed over users, **Scheme B** captures the true maximum and the two schemes perform equivalently. Also notice that the bounds for **Scheme B** tighten over the number of users.

4. ASYMPTOTIC SCALING

Ultimately, the asymptotic performance of these schemes compared with the optimal sum-rate capacity that is achieved via dirty paper coding is of primary interest. In order to utilize the theorem in [1], which shows that a scheme asymptotically achieves the same scaling as the sum-rate capacity, the distributions for the schemes must be shown to satisfy some additional properties. Used in the proof of the optimal asymptotic scaling is a theorem due to Uzgoren ([13]) and a corollary that is proved in the appendix of [1]. First, it must be shown that $F_s(x)$ satisfy the conditions of the theorem and the corollary, which is shown to be true but omitted due to space constaints. With the conditions satisfied, the following theorem is now restated:

Theorem 3. *([1]) Let M and ρ be fixed and $N = 1$. Then*

$$\lim_{n\to\infty} \frac{R}{M \log\log n} = 1$$

where R is the throughput of our scheme.

In the single receive antenna case, i.e. $N = 1$, the value of $M \log\log n$ comes from the fact that the transmitter is selecting the maximum SINR from n values for each beam. For a single receive antenna per user, it is known that the optimal sum-rate capacity scales as $M \log\log n$, and so the scaling is optimal. For **Scheme A** in the more general case where $1 < N \leq M$, the sequence of random variables for each transmit beam seen at the transmitter is $x_1, \ldots, x_{nN}$, thus the dominator in Theorem (4) becomes $M \log\log nN$. The following corollary summarizes the more general case.

Corollary 1. *Let M, $N \leq M$, and ρ be fixed. Then for* ***Scheme A****, the throughput satisfies*

$$\lim_{n\to\infty} \frac{R}{M \log\log nN} = 1.$$

It should briefly be noted that a scaling rate of $M \log\log nN$ is essentially the same rate as $M \log\log n$ since N is a constant that inside the double logarithm becomes inconsequential in the limit as n goes to infinity.

The denominator term in Theorem (3) is determined by the number of observations that the maximum is taken over. **Scheme B** feeds back only the largest SINR measured at each user. The sequence of random variables to be maximized over for each beam is then $x_1, \ldots, x_{M_i}$, where M_i is the number of SINR values fed back for the i^{th} beam. This led to the binomial type expression of the maximum SINR. What is the scaling when the number of terms to be maximized over is random?

It should be noted that when only the largest SINR at a user is fed back, the maximum order statistic over the joint distribution is not known, which led to bounding the joint CDF from above and from below. It can be shown that both bounds in Equation (10) satisfy the requirements for the results of Theorem (3) to hold. Because the joint distribution is continuous and bounded point-wise by two distributions which scale as Theorem (3), the joint distribution of the maximums scales in the same manner.

Theorem (3) is concerned with the asymptotic scaling relative to the sum-rate capacity. Using the weak law of large numbers, as the number of users n grows, the number of values fed back for each transmit beam converges to $\frac{n}{M}$. In **Scheme B** let $Y_{i,n}$ be the sequence of random variables in n denoting the number of SINR values fed back for the i^{th} beam in a system with n users. Let $\theta = \frac{1}{M}$ be the probability that a particular transmit beam is selected. The sequence of random variables $Y_{i,n}$ are binomially distributed with probability of success θ. Define $X_{i,n} = \frac{Y_{i,n}}{n}$. Then this sequence of random variables, by the central limit theorem, satisfies

$$\lim_{n\to\infty} \sqrt{n}\left(X_{i,n} - \frac{1}{M}\right) \to \mathcal{N}\left(0, \frac{1}{M}\left(1-\frac{1}{M}\right)\right) \quad (14)$$

Let $\sigma^2 = \frac{1}{M}\left(1 - \frac{1}{M}\right)$ and define the function $g(x,\theta) = [F_s(x)]^{(nNM\theta)}$. Then by the Delta Method ([14])

$$\sqrt{n}\left[g\left(x, X_{i,n}\right) - g\left(x,\theta\right)\right] \rightarrow \mathcal{N}\left(0, \sigma^2\left[g'\left(x,\theta\right)\right]^2\right) \qquad (15)$$

in distribution. Plugging the values into this equation yields

$$\left[\left(F_s(x)\right)^{nNMX_{i,n}} - \left(F_s(x)\right)^{nN}\right] \rightarrow \mathcal{N}\left(0, nM\left(1 - \frac{1}{M}\right)\left[F_s(x)\right]^{2nN} N^2\left(\log\left(F_s(x)\right)\right)^2\right) \qquad (16)$$

for a particular argument x. Because the distribution $F_s(x) < 1$ for any finite argument, the variance of the above distribution tends to zero as the number of users in the system increases. Therefore, estimating any point of the distribution of the extreme order statistic of the SINR where only the maximum SINR seen at each user is fed back can be approximated by $[F_s(x)]^{nN}$. The exponent of this limiting distribution implies the following corollary.

Corollary 2. *The throughput of* ***Scheme B*** *for fixed M, $N \leq M$, and ρ scales as $M \log\log nN$, or*

$$\lim_{n\rightarrow\infty} \frac{R}{M\log\log nN} = 1.$$

The effect of having the number of users grow to infinity is to provide more observations to take the maximum over. Even in **Scheme B** where the feedback is restricted to solely one SINR value, asymptotically in the number of users, the performance scales at the same rate as **Scheme A**. Intuitively, the probability that a single user is the best for multiple beams becomes smaller as the number of users increases, as is shown in [1].

5. CONCLUSION

In this paper we considered the random beamforming scheme proposed in [1]. The SINR distribution function, required for order statistics, was derived and shown to differ from that in previous work. To further reduce the amount of feedback in the system, each user can feedback the maximum SINR across receive antennas and transmit beams. The analysis of the joint distribution in this case becomes burdensome, causing us to resort to bounding techniques. In addition to the difficulties caused by the complex joint distribution, the transmitter takes the maximum over a random number of quantities, which leads to binomial averaging of the bounds applied to the joint distribution. The difference in throughput of the two schemes decreases as the number of users in the system increase.

Because the underlying SINR distribution is shown to differ from that in [1], the asymptotic properties must be re-derived. It can be shown that the same scaling laws found in [1] hold for the new distribution. The scaling law is also found for the reduced feedback scheme.

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