FEEDBACK REDUCTION IN MIMO BROADCAST CHANNELS WITH LMMSE RECEIVERS

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ABSTRACT

In this paper we analyze the performance of random beamforming schemes in a multi-user Gaussian broadcast channel. Each user will have $N > 1$ receive antennas allowing optimal combining to be performed. To notify the transmitter of its current channel state, each user feeds back their SINR after LMMSE combining. Two feedback schemes are analyzed. The first scheme allows each user to feedback the post-processed SINR for each of the random transmit beams. To analyze this scheme, the distribution of the post-processed SINR is found. The second scheme attempts to limit feedback by allowing each user to feedback only the largest observed post-processed SINR and the index of the associated transmit beam. Using the Fréchet bounds, the throughput of the reduce feedback scheme is bounded and shown to have the same asymptotic scaling properties as the first scheme. Empirically, it is observed that as the number of users in the system increases, the reduced feedback scheme approaches the throughput of the scheme without thresholding.

Index Terms:: multi-user MIMO, broadcast channel, random beamforming, LMMSE receivers

1. INTRODUCTION

The users in a broadcast channel experience varying levels of channel quality. In order to achieve the highest sum-rate throughputs, the transmitter must be aware of the channels that the end users are experiencing. A mechanism by which the transmitter is aware of the channel state information is for each user to feed back the observed channel. Because this represents a large amount of information to feed back, the questions that concern this paper are how to reduce feedback and the tradeoff between reduced feedback and performance. In this work, the random beamforming scheme suggested in [1] is considered. Since each user is allowed multiple receive antennas, to increase the receive SINR, LMMSE receivers will be employed at each users. Because throughput is a function of SINR, if each user feeds back the post-processed SINR for each transmit beam, then the transmitter can select the users that are currently experiencing the best channels for each transmit beam. This is in essence the scheme proposed by [1] extended to LMMSE reception, and is the scheme considered in [2], which only considers the specific case of four transmit antennas and two receive antennas per user.

This paper additionally examines a scheme that further reduces the amount of feedback. Limiting each user to feed back only the largest post-processed SINR value observed amongst the beams and the beam index that generated it reduces the amount of feedback per user to a single SINR value and a single beam index. This is in contrast to the first scheme which allows a post-processed SINR value to be fed back for each transmit beam per user.

The paper is organized as follows. In Section 2, the system model is described. In Section 3 the scheme where each user feeds back the post-processed SINR for each transmit beam is analyzed. The distribution of the post-processed SINR for a general number of transmit antennas and receive antennas per user is derived. Section 4 analyzes a reduced feedback scheme where the receiver feeds back only the maximum observed post-processed SINR value and the associated beam index. Section 5 concludes the paper.

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2. SYSTEM MODEL

A block fading channel model is assumed for the Gaussian broadcast channel. The transmitter has M transmit antennas and there are n receivers, each with $N > 1$ receive antennas. It is further assumed that $n \gg M$ and $N \leq M$. Let $\mathbf{s}(t) \in \mathbb{C}^{M\times 1}$ be the transmitted vector of symbols at time slot t and $\mathbf{y}_i(t) \in \mathbb{C}^{N\times 1}$ be the received symbols at the i^{th} user at time slot t. The following model is used for the input-output relationship between the transmitter and the i^{th} user:

$$\mathbf{y}_i(t) = \sqrt{\rho_i} H_i \mathbf{s}(t) + \mathbf{w}_i(t), \quad i = 1, \ldots, n. \tag{1}$$

$H_i \in \mathbb{C}^{N\times M}$ is the complex channel matrix which is assumed to be known at the receiver, $\mathbf{w}_i \in \mathbb{C}^{N\times 1}$ is the white additive noise, and the elements of H_i and $\mathbf{w}_i$ are i.i.d. complex, circularly symmetric Gaussians with zero mean and unit variance. The transmit power is chosen to be M, i.e. $E\{\mathbf{s}^*\mathbf{s}\} = M$, the SNR at the receiver is $E\{\rho_i |H_i \mathbf{s}|^2\} = M\rho_i$ and ρ_i is the SNR of the i^{th} user. It is assumed that $\rho_i = \rho \; \forall i$.

The transmission scheme, as developed in [1], involves generating M random orthonormal vectors ϕ_m for $m = 1, \ldots, M$, where ϕ_m are isotropically distributed. Let $s_m(t)$ be the m^{th} transmit symbol at time t, then the total transmit signal at time slot t is given by

$$\mathbf{s}(t) = \sum_{m=1}^{M} \phi_m(t) s_m(t). \quad (2)$$

The received signal at the i^{th} user is given by

$$\mathbf{y}_i(t) = \sum_{m=1}^{M} H_i(t)\phi_m(t)s_m(t) + \mathbf{w}_i(t). \quad (3)$$

Since each user in the system has $N > 1$ receive antennas, a more complex receiver structure can be utilized. It is known that the optimal linear receiver is the linear MMSE receiver. Assuming that the i^{th} user knows the quantity $H_i(t)\phi_m(t)$ for all m, using the system model defined in Equations (1) - (3), the SINR after optimal combining for the i^{th} user and the j^{th} transmit beam is given by

$$SINR_{i,j} = \phi_j^* H_i^* \left[H_i \left(\sum_{k \neq j} \phi_k \phi_k^* \right) H_i^* + \frac{2}{\rho} I \right]^{-1} H_i \phi_j \quad (4)$$

where * denotes conjugate transposition and I is the identity matrix.

The two schemes considered in this paper depend upon how much information is fed back to the transmitter. The first scheme considers the scenario where each user feeds back all the observed post-processed SINR values, that is user i feeds back $SINR_{i,j}$ for $j = 1, \ldots, m$ from Equation (4). Call this scheme **Scheme A**. The second scheme considers the scenario where each user feeds back only the largest of the observed post-processed SINR values and the associated beam index. That is, user i feeds back $\max_j SINR_{i,j}$ and $\text{argmax}_j SINR_{i,j}$. Call this scheme **Scheme B**. The throughput and asymptotic scaling of these two schemes are analyzed in the remainder of this paper.

3. FEEDING BACK THE SINR PER BEAM

First consider **Scheme A**. The following approximation to the throughput was shown in [1]:

$$\begin{aligned} R &\approx E\left\{ \sum_{m=1}^{M} \log\left(1 + \max_{i=1,\ldots,n} SINR_{i,m}\right) \right\} \\ &= ME\left\{ \log\left(1 + \max_{i=1,\ldots,n} SINR_{i,m}\right) \right\} \quad (5) \end{aligned}$$

The approximation sign is required because there is a small probability, when there are many users, that the same user is the best for multiple transmit beams. To find the approximate rate requires knowledge of the distribution of the post-processed SINR and the use of order statistics.

The distribution of the post-processed SINR is a special case of the work done in [3] and [4]. Applying the specific system model to these works yields the following theorem.

Theorem 1. *The distribution function of the post-processing SINR given by Equation (4) for the system defined in Equations (1) - (3) is given by*

$$F_{MMSE}(x) =$$
$$1 - \frac{\exp\left(-\frac{x}{\rho}\right)}{(1+x)^{M-1}} \sum_{i=1}^{N} \frac{1 + \sum_{j=1}^{N-i} \binom{M-1}{j} x^j}{(i-1)!} \left(\frac{x}{\rho}\right)^{i-1}.$$

The distribution function for the interference limited regime is analyzed in [5], while the SINR distribution is derived for the case $M = 4$ and $N = 2$ in [2]. Even with closed form expression for the post-processed SINR distribution, no closed form solution to Equation (5) is known to the authors. The rate can be approximated by using numerical methods and the distribution given by Theorem (1).

Along the lines of [1] and [2], the asymptotic scaling of the rate of **Scheme A** as the number of users goes to infinity is of interest. The first step in finding the asymptotic scaling rate is to show that $F_{MMSE}(x)$ given in Theorem (1) has an asymptotic distribution of type 3. It is well known from the theory of order statistics (e.g. [6],[7]) that if a distribution has an asymptotic distribution for the maximal order statistic, that it is one of three types. The type 3 asymptotic distribution, also called the Gumbel type, is given by

$$\Lambda_3(x) = e^{-e^{-x}} \quad -\infty < x < \infty \; . \quad (6)$$

With these definitions established, the following theorem can be stated.

Theorem 2. *([8]) The limiting distribution of the extreme order statistics drawn from the distribution $F_{MMSE}(x)$ is of type 3.*

Using the previous theorem, the scaling rate can be shown to satisfy the following corollary:

Corollary 1. *([8]) For fixed M, $N \leq M$, and ρ, the throughput for **Scheme C** asymptotically scales as $M \log\log n$, or*

$$\lim_{n\to\infty} \frac{R}{M \log\log n} = 1.$$

The corollary shows that the asymptotic scaling rate of **Scheme A** is $M \log\log n$, which is the same scaling rate as that found in [1] which did not use LMMSE receivers. While the asymptotic scaling rate is the same, the LMMSE receivers support a higher rate leading to a rate offset between using LMMSE receivers and not using optimal combining.

4. FEEDING BACK THE MAXIMUM SINR PER USER

Scheme B has each user feedback the maximum post-processed SINR value and the beam index that produced it. This scheme is suboptimal in that the global maximum for a particular beam may not be fed back due to the constraint that each user can only feed back one value. Even though it is suboptimal, it will be shown that **Scheme B** has the same asymptotic scaling as **Scheme A**, and empirically it will be shown that the throughput for **Scheme A** and **Scheme B** converge. The analysis of **Scheme B** raises some issues that do not occur in the analysis of **Scheme A**.

The first problem is how to characterize $\max_j SINR_{i,j}$. The main concern is that although the SINR values at a particular user are marginally identically distributed, they are not independent. Thus classical order statistics cannot be utilized. Ideally, the joint distribution could be found, but this is too complicated. To attack the problem, the distribution function will be bounded from above and below. The desired asymptotic results will be shown to hold on the bounds, thus guaranteeing that the true distribution has the desired properties. The bounds of interest are the classical Fréchet Bounds

Theorem 3. *([7], The Fréchet Bounds) Let $F(\mathbf{x})$ be an m-dimensional distribution function with marginals $F_j(x)$, $1 \leq j \leq m$. Then, for all $x_1, x_2, \ldots, x_m$*

$$\max\left(0, \sum_{j=1}^{m} F_j\left(x_j\right) - m + 1\right) \leq F\left(x_1, x_2, \ldots, x_m\right) \leq \min\left(F_1\left(x_1\right), \ldots, F_m\left(x_m\right)\right) \quad .$$

The Fréchet Bounds provide control of the unknown joint distribution of the SINR values after optimal combining, and can provide bounds on the order statistic because all the marginal distributions are identical:

$$\begin{aligned} \max\left(0, M\left(F_{MMSE}\left(x\right) - 1\right) + 1\right) &\leq F_{MaxMMSE}(x) \\ &\leq F_{MMSE}(x) \end{aligned} \quad (7)$$

where $F_{MaxMMSE}$ is the unknown distribution of the maximum SINR at a user after optimal combining. The upper bound in Equation (7) is in terms of $F_{MMSE}(x)$ and thus by Corollary (1) is of type 3. Substituting the distribution into the lower bound and using limiting arguments detailed in [8] shows that the lower bound is also of type 3. Since both the upper and lower bound are of type 3, the unknown distribution $F_{MaxMMSE}$ is also of type 3.

The second issue that arises with **Scheme B** is that if each user only feeds back the maximum SINR and the associated beam index, the number of values to maximize over for each beam at the transmitter is a random number. For example, in Figure 1, there are nine users and at the transmitter, the number of users feeding back a particular beam index is random.

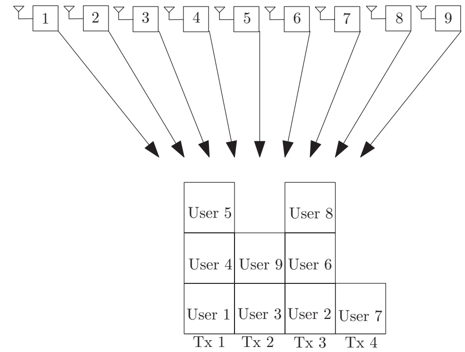

Fig. 1. 9 User System only Feeding Back Beam Index and Largest SINR Over Transmit Beams and Receive Antennas

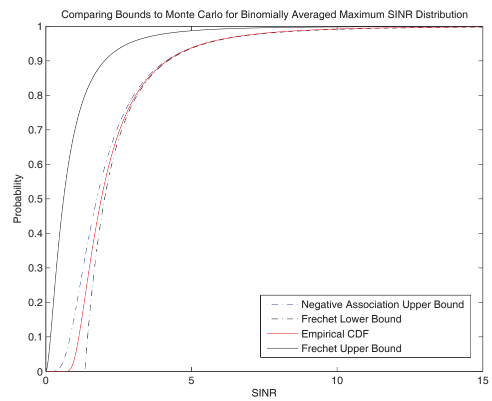

Fig. 2. Theoretical vs. Empirical Distribution for SINR after Optimal Combining for M = 5, N = 2, SNR = 10dB

In this case three users feed back beam index 1, two users feed back beam index 2, and so on.

The number of values to maximize over at the transmitter determines the exponent of the distribution of the order statistics. By the symmetry of the system model, each beam is equally likely to produce the maximum. Thus the number of values to maximize over at the transmitter is determined by the multinomial distribution. As the number of users in the system grows, by the law of large numbers the number of indices fed back for each transmit beam approaches $\frac{n}{M}$. This is made rigorous in [8]. The key is that the exponent grows linearly with n, which combined with the Fréchet bounds yields the following corollary:

Corollary 2. *[8] For fixed M, $N \leq M$, and ρ, the throughput for **Scheme D** asymptotically scales as $M \log\log n$, or*

$$\lim_{n\to\infty} \frac{R}{M \log\log n} = 1.$$

Figure 2 shows the empirical distribution distribution of the maximum SINR at a user as well as the Fréchet bounds. Although the Fréchet bounds are sufficient to show the asymptotic scaling, the upper Fréchet bound is very loose. For numerical results, a tighter tractable bound is useful. Although it could not be shown rigorously, empirical evidence suggests that the SINR values after optimal combining are negatively associated. The definition of negative association is as follows:

Definition 1. *([9], Negative Association) Random variables* $X_1, X_2, \ldots, X_k$ *are said to be negatively associated if for every pair of disjoint subsets* A_1, A_2 *of* $\{1, 2, \ldots, k\}$,

$$Cov\left\{f_1\left(X_i, i \in A_1\right), f_2\left(X_j, j \in A_2\right)\right\} \leq 0$$

whenever f_1 *and* f_2 *are both increasing or both decreasing.*

Negative association is a multivariate generalization of negative correlation. Intuitively, negative association seems plausible because the larger one signal's power, the more interference it can cause, thus the other signals' SINRs are most likely going to decrease. The key property of interest is that if $X_1, \ldots, X_N$ are negatively associated random variables, then

$$\Pr\left(X_1 \leq x_1, \ldots, X_N \leq x_N\right) \leq \prod_{i=1}^{N} \Pr\left(X_i \leq x_i\right).$$

Applying this to our distribution, we conjecture a tighter upper bound is given by

$$F_{MaxMMSE}(x) \leq \left[F_{MMSE}(x)\right]^M \qquad (8)$$

The bound provided by Equation (8) is much tighter than the upper Fréchet bound as noticed in Figure 2. Using the lower Fréchet bound and the upper bound provided by negative association, Figure 3 compares the throughput of **Scheme A** and **Scheme B** as a function of the number of users in the system for various SNRs. As the number of users in the system increases, the two schemes converge to the same throughput. This is expected, since as the number of users increases, the probability of not feeding back the global maximum for a particular transmit beam decreases, thus yielding equal performance with less feedback. It is also observed that as the SNR increases, the system becomes dominated by interference. For example, in Figure 3, the difference in throughput between an SNR of 20 dB and 30 dB is negligible.

5. CONCLUSION

Receivers with multiple receive antennas are capable of performing more sophisticated reception techniques than receivers with a single antenna. It is well known that the optimal linear receiver is the LMMSE receiver. Extending the seminal work of [1], a random beamforming scheme where each user feeds back the post-processed SINR is considered and shown

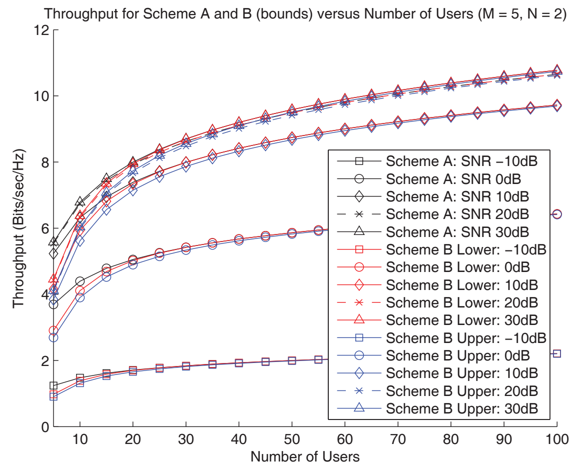

Fig. 3. Throughput as a Function of Number of Users for Different Schemes for M = 5, N = 2 and Various SNRs

to scale as $M \log\log n$. Feedback reduction is possible by considering feeding back only the maximum post-processed SINR and the associated beam index. This reduced feedback scheme also scales as $M \log\log n$, and it is observed that as the number of users increases, the reduced feedback scheme approaches the performance of the un-thresholded scheme.

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