The Proportional Fair Sharing Algorithm Under i.i.d. Models

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***Abstract*—The proportional fair sharing (PFS) algorithm has been used in multi-user systems as an attempt to balance fairness and performance of the system throughput. Motivated by the cellular downlink scheduling problem, it is shown that when the rates of each user are i.i.d., the performance of the PFS scheduling algorithms is asymptotically equivalent to a purely greedy scheduling algorithm. The mean asymptotic throughput of the PFS algorithm is characterized and the rate of convergence to this limit is derived under i.i.d. models. Additionally the asymptotic covariance matrix about the convergence point is stated.**

I. Introduction

Scheduling users efficiently in the downlink channel is an important question whose solutions balance system throughput and fairness. Suppose there are n users in a given cell and the base station has M transmit antennas and $n > M$. When there is no coding across time or frequency, at most a subset of M of the n users can be scheduled for simultaneous transmission by the base station if all the users are to receive independent signals. The central question addressed by any scheduling algorithm is how to select the $k \leq M$ users for transmission during any given scheduling epoch.

Greedy scheduling algorithms are attractive due to their ease of implementation and good performance but ignores any fairness criterion. The proportional fair sharing (PFS) algorithm is a scheduling algorithm that tries to balance performance and fairness by considering the ratio of each user's current rate to their average scheduled rate and selecting the user with the largest ratio.

This contribution shows that when the rates are i.i.d., under suitable conditions on the distribution, the PFS algorithm's performance converges to the performance of the greedy algorithm asymptotically in time. A transmit scheme such that the rates are captured in a scalar sufficient statistic that is measured at each receiver such as signal-to-noise ratio (SNR) or signal-to-interference plus noise ratio (SINR) is assumed. In the appropriate state space, the convergence is equivalent to showing the two algorithms converge to the same stationary limit point. The rate of convergence of the PFS algorithm to the limit point is characterized. Therefore the PFS algorithm can be used to ensure some degree of fairness and asymptotically achieve the performance of the greedy algorithm. These results are shown by leveraging the theory of stochastic approximations [1], as is done in the motivating work [2].

The organization of the paper is as follows: the PFS algorithm is described in detail in Section II. The convergence of the PFS algorithm to the greedy algorithm when the rates are bounded is considered in Section III and the convergence results are extended to the case where the rates are unbounded in Section IV. The rate of convergence to the fixed stationary point and the asymptotic covariance matrix of the distance from the stationary point is stated in V. Section VI summarizes the results and concludes the contribution.

II. The Proportional Fair Sharing Algorithm

The general proportional fair sharing problem formulation follows the set-up established in [2]. The results of [2] show that under general conditions the PFS algorithm converges, but does not address where it converges to due to the generality of the conditions. Additionally, due to the generality of the conditions, the convergence is weak. In this work, we show that under an i.i.d. model, the convergence is to the performance of the greedy algorithm almost surely.

Let there be N users in the system and let $X_{i,n}$ be the instantaneous rate of user i at time slot n. The $X_{i,n}$ are assumed to be i.i.d. continuous, non-negative random variables for all $1 \leq i \leq N$ and $n \in \mathbb{N}^+$. At time $n+1$, let the rates $X_{i,n+1}$ be known at the scheduler for all i. The indicator random variable $I_{i,n+1}$ denotes the event that user i is scheduled for time slot $n+1$. The average scheduled throughput of the i^{th} user after n time slots is denoted by $\theta_{i,n}$ which is defined as follows:

$$\theta_{i,n} = \sum_{l=1}^{n} \frac{X_{i,l} I_{i,l}}{n}. \tag{1}$$

The greedy algorithm selects the user with the largest instantaneous throughput at every time slot n, i.e. the scheduling function for time slot $n+1$ is given by

$$\arg \max_{1 \leq i \leq N} \{X_{i,n+1}\}. \tag{2}$$

The PFS algorithm selects the user with the largest ratio

$$\arg \max_{1 \leq i \leq N} \left\{ \frac{X_{i,n+1}}{\theta_{i,n}} \right\}. \tag{3}$$

When the PFS algorithm starts, all the initial average scheduled rates $\theta_{i,n}$ are zero, and the ratio in Equation 3 is undefined. This was observed in [2] and a solution is to modify the ratio slightly. Let d_i be positive constants for

$1 \leq i \leq N$, which can be arbitrarily small. Define the modified PFS algorithm as performing the following function:

$$\arg \max_{1 \leq i \leq N} \left\{ \frac{X_{i,n+1}}{d_i + \theta_{i,n}} \right\}. \tag{4}$$

The ratio defined in Equation 4 is always well defined since the rates are non-negative. In this work it is assumed that $d_i = d$ for all i and that d is incredibly small, and is considered only to be needed so that the ratio is well defined during the initial phases of the algorithm, i.e. until every user is scheduled at least once so that average throughput term is non-zero for every user. Because the rates $X_{i,n}$ are assumed to be continuous, the probability of there being a tie in either Equation 3 or 4 is zero. Having established the basics of the PFS algorithm, its convergence properties will now be addressed.

III. Convergence with Bounded Rates

In this section the non-negative i.i.d. rate random variables $X_{i,n}$ are assumed to be bounded from above, i.e. there exists some constant K such that $\Pr(X_{i,n} > K) = 0$ for all i and n. The key to showing the convergence result is to put the PFS algorithm into the frame work of a stochastic approximation (SA) algorithm, and leverage the theory of SA algorithms. The general form of the SA algorithm, as defined in [3], is as follows. Let $(\omega^n)_{n\geq 1}$, $(\eta^n)_{n\geq 1}$ be two sequences of $\mathbb{R}^p$ and $\mathbb{R}^d$ valued vectors respectively and let $H : \mathbb{R}^d \times \mathbb{R}^p \mapsto \mathbb{R}^d$. The stochastic approximation algorithm is defined by the sequence $(\theta^n)_{n\geq 1}$ given as follows:

$$\begin{cases} \theta^0 \in \mathbb{R}^d \\ \theta^{n+1} = \theta^n + \epsilon_{n+1} H\left(\theta^n, \omega^{n+1}\right) + \epsilon_{n+1}\eta^{n+1}, n \in \mathbb{N}^+, \end{cases} \tag{5}$$

where $(\epsilon_n)_{n\geq 1}$ is a sequence of positive real numbers called the gain of the algorithm and η^n is a small residual perturbation.

Let $\theta^n = [\theta_{1,n}, \ldots, \theta_{N,n}]^T \in \mathbb{R}^N$ be the vector of average throughputs, where it is understood that superscripts are time indices for vector quantities and subscripts indicate elements of a vector at a particular time index. Likewise, let $X^n = [X_{1,n}, \ldots, X_{N,n}]^T \in \mathbb{R}^N$ be the vector of rates and let $I^{n+1}(\theta^n, X^{n+1}) \in \mathbb{R}^{N\times N}$ be the indicator matrix which is an $N \times N$ matrix of zeros except for the i^{th} diagonal element which is unity if user i is scheduled for time slot $n+1$. The indicator matrix $I^{n+1}(\theta^n, X^{n+1})$ is explicitly written as a function of the vector of average throughputs θ^n and current rates X^{n+1} since these vectors determine which component is non-zero by Equation 4.

The vector of average throughputs θ^{n+1} will now be written in the SA framework of Equation 5. Let $\omega^{n+1} = X^{n+1}$, $\epsilon_{n+1} = \frac{1}{n+1}$, and the perturbation sequence $\eta^{n+1} = 0$ for all n. Let it also be assumed that the initial average throughputs are zero for every user, i.e. $\theta^0 = \mathbf{0} \in \mathbb{R}^d$. Then the following equation gives the desired recursive relationship:

$$\begin{aligned} \theta^{n+1} &= \theta^n + \epsilon_{n+1} H\left(\theta^n, X^{n+1}\right) \\ &= \theta^n + \epsilon_{n+1}\left[I^{n+1}(\theta^n, X^{n+1})X^{n+1} - \theta^n\right] \\ &= \theta^n + \epsilon_{n+1} Y^n, \end{aligned} \tag{6}$$

where $H\left(\theta^n, X^{n+1}\right) = \left[I^{n+1}(\theta^n, X^{n+1})X^{n+1} - \theta^n\right] = Y^n$. Equation 6 is established in [2].

The ODE Method for solving SA algorithms attempts to show that the random SA algorithm generally behaves like a dynamical system, and that asymptotically the SA algorithm converges to a stationary point or stationary set of a set of differential equations. Let h be called the *mean function*, and in the i.i.d. setting, as mentioned in [3], it can be defined as:

$$h(\theta) \triangleq \int H(\theta, x)\mu(dx) \tag{7}$$

where $X \in \mathbb{R}^N$ have common distribution μ, $\theta \in \mathbb{R}^N$ and $H(\theta, \cdot)$ is μ-integrable for every $\theta \in \mathbb{R}^N$. The interpretation of the mean function is that given the current location in the θ-space is θ, on average one will move in the direction $h(\theta)$. Let $(\mathcal{F}_n)_{n\geq 0}$ be the filtration generated by θ^0 and $(X^n)_{n\geq 1}$. Then letting the random vector of the current average throughputs θ^n be the argument of the mean function produces

$$h(\theta^n) = \mathbb{E}\left(H(\theta^n, X^{n+1})|\mathcal{F}_n\right). \tag{8}$$

The right hand side of Equation 8 says where the algorithm is going to move on average in the next time step given the past (the information contained in the filtration $\mathcal{F}_n$). Lastly, define a martingale difference sequence (MDS) as follows:

Definition 1: Let $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 1}$ be two $\mathbb{R}^d$-valued random sequences and let $(\mathcal{F}^n)_{n\geq 1}$ be the filtration generated by $(X^n)_{n\geq 1}$. The sequence $(Y^n)_{n\geq 1}$ is called a martingale difference sequence if

$$\mathbb{E}(Y^{n+1}|\mathcal{F}^n) = 0 \text{ for all } n \in \mathbb{N}.$$

Having established the preceding definitions, Equation 6 can be written in the following form suitable for analysis by the ODE method:

$$\begin{aligned} \theta^{n+1} =& \theta^n + \epsilon_{n+1}h(\theta^n) + \\ & \epsilon_{n+1}\left(I^{n+1}(\theta^n, X^{n+1})X^{n+1} - \theta^n - h(\theta^n)\right) \quad (9) \\ =& \theta^n + \epsilon_{n+1}h(\theta^n) + \epsilon_{n+1}\delta M^{n+1} \quad (10) \end{aligned}$$

where $h(\theta^n)$ is the mean function and δM^{n+1} is a MDS. That δM^{n+1} is a MDS follows from the definition of H, mean function h (Equation 8) and using the filtration $(\mathcal{F}^n)_{n\geq 0}$ generated by θ^0 and $(X^n)_{n\geq 1}$.

The fact that Equation 6 could be written in the form of the sum of a mean function and a MDS is not surprising. As pointed out in Chapter 5 of [4], this form naturally arises when Y^n can be written in the form $H(\theta^n, X^n)$ and the X^n are mutually independent, as is the case here. If the conditions of the celebrated Kushner-Clark Theorem ([3], [4]) can be verified, the result of the theorem says that almost surely the limit of sample paths θ^n are trajectories of the differential equation

$$\dot{\theta} = h(\theta) \tag{11}$$

The key implication of the Kushner-Clark Theorem is that the SA algorithm, and thus the PFS algorithm, can asymptotically be analyzed using the theory of differential equations and dynamical systems. Additionally, if it can be shown that there is a unique singleton $\{\theta^*\}$ such that regardless of the initial condition θ^0, the zero or equilibrium point of Equation 11 is θ^*, then the Kushner-Clark Theorem also says that $(\theta^n)_{n\geq 1}$ converges almost surely to θ^* regardless of the initial condition.

Having established the stochastic approximation framework, it will now be shown that under bounded non-negative i.i.d. random variables the average rate vector θ^n of the PFS algorithm converges to the average rate vector $\tilde{\theta}^n$ of the greedy algorithm. This result will be shown via the following steps. First, due to space considerations it will be assumed that the conditions required of the Kushner-Clark Theorem hold so the solution of the SA algorithm converges to an equilibrium point of the differential equation given in Equation 11. Next, a theorem in [2] is leveraged that states there exist a unique equilibrium point, and thus the SA algorithm converges almost surely to θ^*. Lastly, it will be shown that θ^* corresponds to the convergence point $\tilde{\theta}^*$ under the greedy algorithm, so the two algorithms are asymptotically equivalent. It should be pointed out that under the i.i.d. model which yields Equation 9, almost sure convergence results are attainable, as opposed to the weak convergence results found in [2] which consider more general random sequences $(X^n)_{n\geq 1}$.

The existence of a unique global attractor θ^* for Equation 11 is given by Theorem 2.2 in [2]. The proof of the theorem is based on dynamical systems theory, more specifically the monotonicity property of the solution to Equation 11. Combining the result of the Kushner-Clark Theorem with Theorem 2.2 in [2], one obtains that the PFS algorithm converges almost surely to a unique equilibrium point θ^* regardless of the initial condition θ^0.

Since it is known that the PFS algorithm converges to a unique θ^* almost surely, the last step is to show that θ^* corresponds to the average throughput given by the greedy algorithm. First, the average throughput of the greedy algorithm under the i.i.d. model is established in the following lemma:

Lemma 1: Let $X_{1,n}, \ldots, X_{N,n}$ be i.i.d. non-negative bounded random variables for all $n \in \mathbb{N}^+$. Define the maximum order statistic at time n as $M_{N,n} = \max\{X_{1,n}, \ldots, X_{N,n}\}$ and let $\tilde{\theta}_{i,n}$ be the average throughput of user i at time n under the greedy scheduling algorithm given by Equation 2. Then the average throughput of the greedy algorithm $\tilde{\theta}_{i,n}$ converges to $\tilde{\theta}_i^* = \frac{\mathbb{E}[M_N]}{N}$ almost surely for all $i \in \{1, \ldots, N\}$ and M_N does not depend on time.

Proof: Let $\tilde{I}_{i,n}$ be the indicator random variable for the event that user i was scheduled at time slot n under the greedy algorithm. By Equation 2, $\tilde{I}_{i,n}(X^n) = 1$ if and only if $\max\{X_{1,n}, \ldots, X_{N,n}\} = X_{i,n}$, and $\tilde{I}_{i,n}(X^n) = 0$ otherwise. Because the algorithm is greedy, $\tilde{I}_{i,n}(X^n)$ is only a function of the current rates X^n, and for simplicity the the explicit functional dependence will be dropped from the notation. Due to the i.i.d. $X_{i,n}$s, from [5] it is known that $\Pr(\tilde{I}_{i,n}(X^n) = 1) = \frac{1}{N}$ for all i and all n. The average throughput of user i at time n can be written

$$\tilde{\theta}_{i,n} = \frac{1}{n}\sum_{j=1}^{n} \tilde{I}_{i,j} X_{i,j}. \tag{12}$$

The expectation of $\tilde{I}_{i,j} X_{i,j}$ is given by

$$\mathbb{E}[\tilde{I}_{i,j} X_{i,j}] = \mathbb{E}[X_{i,j} | \tilde{I}_{i,j} = 1] \cdot \Pr(\tilde{I}_{i,j} = 1) = \frac{\mathbb{E}[M_{N,n}]}{N}. \tag{13}$$

By the underlying i.i.d. assumption, $\mathbb{E}[M_{N,n}]$ does not depend on the time index n, so it will be dropped. Let $Z_{i,n} = \tilde{I}_{i,j} X_{i,j}$, then Equation 12 can be rewritten as $\tilde{\theta}_{i,n} = \frac{1}{n}\sum_{j=1}^{n} Z_{i,j}$, and the $Z_{i,j}$ are i.i.d. for each i. By the boundedness and non-negativity of the $X_{i,n}$

$$\mathbb{E}[|Z_{i,n}|] = \mathbb{E}[Z_{i,n}] = \frac{\mathbb{E}[M_N]}{N} < \infty,$$

so the Strong Law of Large Numbers (SLLN) can be applied and yields

$$\tilde{\theta}_{i,n} \to \tilde{\theta}_i^* = \frac{\mathbb{E}[M_N]}{N} \text{ a.s. as } n \to \infty \text{ for all } i \in \{1, \ldots, N\}. \tag{14}$$

■

With the previous lemma established, the convergence of the PFS algorithm to the greedy algorithm is proven in the following theorem:

Theorem 1: Let $X_{1,n}, \ldots, X_{N,n}$ be i.i.d. non-negative bounded random variables for all $n \in \mathbb{N}^+$. The PFS algorithm given by Equation 4 converges to the same average throughput point $\tilde{\theta}^*$ as the greedy scheduling algorithm given by Equation 2. Therefore the asymptotic performance in time of the two algorithms under i.i.d. rates is identical.

Proof: Due to the Kushner-Clark Theorem and Theorem 2.2 from [2], the PFS algorithm converges to a unique point θ^*. This solution θ^* is the unique zero of the mean function h. Lemma 1 states that the greedy algorithm under i.i.d. rates converges to the average throughput point $\tilde{\theta}^* = \frac{\mathbb{E}[M_N]}{N} \cdot \mathbf{1}_N \in \mathbb{R}^N$, where $\mathbf{1}_N$ is the N-dimensional vector of all ones. The last step to show convergence of the PFS algorithm to the greedy algorithm is to show that the convergence point of the greedy algorithm $\tilde{\theta}^*$ is a zero of the mean function h:

$$h\left(\tilde{\theta}^*\right) = \int H\left(\tilde{\theta}^*, x\right) \mu(dx) \tag{15}$$

$$= \mathbb{E}_X\left[I\left(\tilde{\theta}^*, X\right) X\right] - \tilde{\theta}^* \tag{16}$$

$$= \frac{\mathbb{E}[M_N]}{N} \cdot \mathbf{1}_N - \tilde{\theta}^* \tag{17}$$

$$= 0. \tag{18}$$

Therefore, $\tilde{\theta}^*$ is a zero of the mean function h, completing the proof. ■

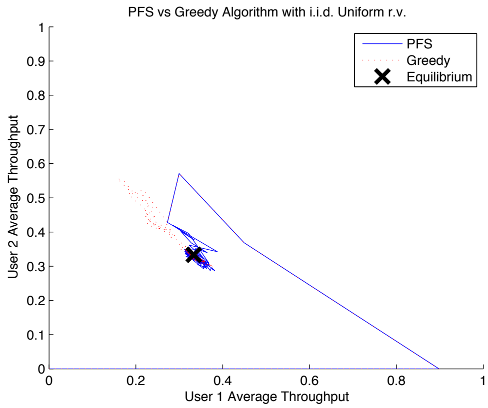

Fig. 1. Sample Paths of the PFS and Greedy Algorithms for 2 i.i.d. Uniform Rates

Figure 1 shows the sample paths of the PFS algorithm and the greedy algorithm on the same set of realized i.i.d. uniform random variables. For the two-dimensional uniform case, the mean of the maximum order statistic is $2/3$, so the equilibrium point is $[1/3, 1/3]^T$. As is expected by the preceding theory, both the PFS and greedy algorithms converge to the equilibrium point.

IV. Convergence with Unbounded Rates

Scheduling in wireless systems is often based on a metric which is theoretically unbounded. For example, in [6], the scheduling decision is based on the SINR random variables which are supported on the non-negative real line. Unbounded random variables pose problems for the SA algorithm because at any iteration, the algorithm may move an arbitrarily large distance and prevent convergence. In this section, conditions on the unbounded $X_{i,n}$ are found such the SA algorithm converges to the greedy algorithm.

The question that is first addressed is when does the greedy algorithm converge? The necessary and sufficient conditions for the convergence of the greedy algorithm under the i.i.d. model are given by the following theorem:

Theorem 2: Let $X_{1,n}, \dots, X_{N,n}$ be i.i.d. non-negative random variables for all $n \in \mathbb{N}^+$, let $\tilde{\theta}_{i,n}$ be the average throughput of user i at time n under the greedy scheduling algorithm given by Equation 2 and $M_{N,n} = \max\{X_{1,n}, \dots, X_{N,n}\}$. A necessary and sufficient condition for the convergence of the greedy algorithm is that $\mathbb{E}[X_{i,n}] < \infty$, in which case for each user $i \in \{1, \dots, N\}$ the algorithm converges to $\tilde{\theta}_i^* = \frac{\mathbb{E}[M_N]}{N} < \infty$.

Proof: First, necessary and sufficient conditions on the existence of the mean of the maximum order statistic must be given if it is hoped that the algorithm converge to the desired quantity. This is given by the following Lemma:

Lemma 2: Let $X_1, \dots, X_N$ be i.i.d. non-negative random variables and $M_N = \bigvee_{i=1}^N X_i$. Then $\mathbb{E}[M_N] < \infty$ if and only if $\mathbb{E}[X_i] < \infty$.

To provide necessary conditions on the convergence of the greedy algorithm, certain conditions such that the normalized sum of i.i.d. random variables diverges will be needed. These conditions are provided by the following theorem:

Theorem 3: (Thm 7.2, Chapter 1, [7])
Let $X_1, X_2, \dots$ be i.i.d. with $\mathbb{E}X_i^+ = \infty$ and $\mathbb{E}X_i^- < \infty$ where X_i^+ and X_i^- are the positive and negative parts of the random variables respectively. If $S_n = X_1 + \cdots + X_n$ then $S_n/n \to \infty$ almost surely.
With the result of Lemma 2 and Theorem 3 stated, Theorem 4 can be proven. By Lemma 2, a necessary and sufficient condition for the existence of $\mathbb{E}[M_{N,n}]$ is that $\mathbb{E}[X_{i,n}] < \infty$. Thus $\mathbb{E}[X_{i,n}] < \infty$ is a sufficient condition for the convergence of the greedy algorithm because the hypotheses of the SLLN hold and using the same arguments as Lemma 1. Because $\mathbb{E}[X_{i,n}^-] = \mathbb{E}[M_{N,n}^-] = 0$, if $\mathbb{E}[X_{i,n}] = \infty$, then by Lemma 2 $\mathbb{E}[M_{N,n}] = \infty$ and by Theorem 3 the greedy algorithm would diverge to infinity. Therefore the condition $\mathbb{E}[X_{i,n}] < \infty$ is a necessary condition. When the greedy algorithm converges, by the SLLN it converges to $\tilde{\theta}_i^* = \frac{\mathbb{E}[M_N]}{N}$ for each user i, so the convergence point is $\tilde{\theta}^* = \frac{\mathbb{E}[M_N]}{N}\mathbf{1}_N$. ■

The next step is to show that these same conditions guarantee the convergence of the PFS algorithm to the same point $\tilde{\theta}^* = \frac{\mathbb{E}[M_N]}{N}\mathbf{1}_N$. A method of analysis for the convergence of SA algorithms with possibly unbounded increments is the method of expanding truncations.

The expanding truncation method is based on the following formulation of the PFS algorithm:

$$\begin{aligned}\theta^{n+1} =& I_{[\|\theta^n+\epsilon_{n+1}Y^n\|<b]}\left(\theta^n + \epsilon_{n+1}Y^n\right) + \\ & I_{[\|\theta^n+\epsilon_{n+1}Y^n\|\geq b]}\theta^\dagger\end{aligned} \tag{19}$$

where $\theta^\dagger \in \mathbb{R}^N$ is a fixed point. Here $I_{[\cdot]} \in \mathbb{R}^{N\times N}$ is an indicator matrix of zeros whose diagonal elements are one when the argument in the subscript is true, and zero otherwise. Equation 19 states that when the norm of the next iteration $\theta^n + \epsilon_{n+1}Y^n$ is less than some positive constant b, update the state θ^{n+1} as normal, otherwise the state vector gets updated as $\theta^{n+1} = \theta^\dagger$. Equation 19 is formulated to prevent the state variable from leaving some bounded set in $\mathbb{R}^N$.

The basic idea is that the bounded set containing the SA algorithm is expanded at each iteration, and if it can be shown that the SA algorithm leaves these bounded sets only a finite number of times, then the performance of the expanding truncation method is asymptotically equivalent to the SA algorithm as if it were unconstrained. The theorems provided in [8] that show convergence of the expanding truncations methods typically depend on properties of the mean function h. In this case, the mean function h is not explicitly known in closed form. Because of the similarity of the PFS algorithm with i.i.d. rates to the Strong Law of Large Numbers (SLLN), Etemadi's proof [9] of the SLLN can be emulated to prove the convergence of the PFS algorithm with unbounded rates under the same conditions as Theorem 2 to the greedy algorithm. The following theorem, given without proof, provides this convergence.

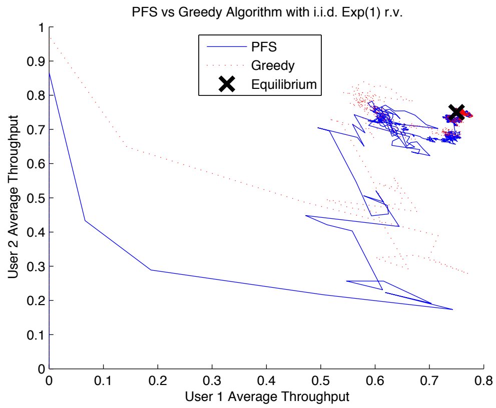

Fig. 2. Sample Paths of the PFS and Greedy Algorithms for 2 i.i.d. Exp(1) Rates

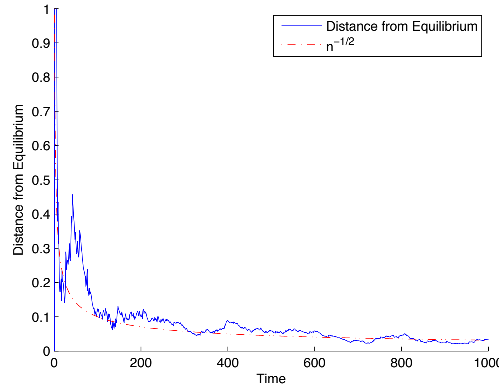

Fig. 3. Distance from Equilibrium versus Time for $N = 3$, $X_{i,n} \sim$ Exp(1)

Theorem 4: Let $X_{1,n}, \ldots, X_{N,n}$ be i.i.d. non-negative random variables such that $\mathbb{E}[X_{i,n}] < \infty$ for all $n \in \mathbb{N}^+$. The PFS algorithm given by Equation 4 converges to the same average throughput point $\tilde{\theta}^* = \frac{\mathbb{E}[M_N]}{N} \cdot \mathbf{1}_N$ as the greedy scheduling algorithm given by Equation 2 and Theorem 2.

This section is concluded with an example. Figure 2 shows the sample paths of the PFS and greedy algorithm on the same set of realized i.i.d. exponential-1 random variables. The exponential random variables have unbounded support, but the convergence still occurs as expected. In the case of two exponential random variables, the mean of the maximum order statistic is 1.5, so the equilibrium point is given by $[3/4, 3/4]^T$.

V. Rate of Convergence

In Sections III and IV, it is shown that the PFS algorithm under the i.i.d. model converges to a single average rate vector θ^*, and that θ^* corresponds to the performance of the greedy scheduler. The goal of this section is to find out how fast $\|\theta^n - \theta^*\| \to 0$. This section will leverage the results in Chapter 3 of [8], specifically the following theorem:

Theorem 5: (Theorem 3.1.1, [8]) Under suitable conditions (see Chapter 3 of [8]) , θ^n as given in Equation 19 converges to θ^* with the following convergence rate:

$$\|\theta^n - \theta^*\| = o\left(\epsilon_n^{\beta}\right), \tag{20}$$

for some $\beta \in (0, 1]$.

It can be shown that the conditions required to satisfy Theorem 5 are satisfied for the problem of interest. Once these conditions are verified, the following corollary can be proven:

Corollary 1: Let $X_{i,n}, \ldots, X_{N,n}$ be continuous, non-negative i.i.d. random variables for $n \in \mathbb{N}^+$ with continuous densities f such that $\mathbb{E}[X_{i,n}^2] < \infty$. The convergence rate of the PFS algorithm is given by

$$\|\theta^n - \theta^*\| = o\left(\frac{1}{n^{\beta}}\right), \tag{21}$$

where $\beta = \frac{1}{2} - \epsilon$ and $\epsilon > 0$.

Figure 3 shows the results of a simulation run with $N = 3$ and $X_{i,n} \sim$ Exp(1). The convergence towards equilibrium of the PFS algorithm will not be monotonic due to the randomness of the rates and sample paths of the algorithm. The results of Theorem 5 and Corollary 1 give the asymptotic scaling rate in terms of o notation. The dashed curve in Figure 3 shows the function $f(n) = n^{-\frac{1}{2}}$. The long term convergence rate of the PFS algorithm to equilibrium is slightly slower than this, but the curve shows that the results of the theory are reasonable.

VI. Conclusion

This contribution addresses the asymptotic performance of the proportional fair sharing algorithm under i.i.d. rate models. Under this model it is shown in the state space of average throughput and under suitable conditions on the distribution of the rates that the PFS algorithm converges to the same fixed point as the greedy scheduling algorithm. Therefore, from the point of view of average throughput, asymptotically in time there is no difference between using the PFS algorithm and the greedy algorithm. Under i.i.d. models, it is shown that the rate of convergence of the PFS algorithm to the equilibrium average throughput point is $o\left(\frac{1}{n^{\beta}}\right)$ where $\beta = \frac{1}{2} - \epsilon$ and $\epsilon > 0$.

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