Reduced Feedback Schemes Using Random Beamforming in MIMO Broadcast Channels

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***Abstract*—A random beamforming scheme for the Gaussian MIMO broadcast channel with channel quality feedback is investigated and extended. Considering the case where the n receivers each have N receive antennas, the effects of feeding back various amounts of signal-to-interference-plus-noise ratio (SINR) information are analyzed. Using the results from order statistics of the ratio of a linear combination of exponential random variables, the distribution function of the maximum order statistic of the SINR observed at the receiver is found. The analysis from viewing each antenna as an individual user is extended to allow combining at the receivers, where it is known that the linear MMSE combiner is the optimal linear receiver and the CDF for the SINR after optimal combining is derived. Analytically, using the Delta Method, the asymptotic distribution of the maximum order statistic of the SINR with and without combining is shown to be, in the nomenclature of extreme order statistics, of type 3. The throughput of the feedback schemes are shown to exhibit optimal scaling asymptotically in the number of users. Finally, to further reduce the amount of feedback, a hard threshold is applied to the SINR feedback. The amount of feedback saved by implementing a hard threshold is determined and the effect on the system throughput is analyzed and bounded.**

***Index Terms*— Broadcast channel, channel state information (CSI), multiuser diversity, order statistics.**

I. Introduction

The users in a broadcast channel experience varying levels of channel quality. For high throughput, it is useful for the transmitter to be fully aware of the channel to all the users. This represents a large amount of information that must be known at the transmitter, especially if the number of users n is large. A mechanism by which the transmitter is aware of the channel state information is for each user to feed back the observed channel. To send back the observed channel may impose unreasonable complexity, so schemes wherein the users send back partial channel information, but still realize the major benefits of multiuser diversity, are of interest. The issues that concern this paper are how to reduce the amount of feedback, the tradeoff between reduced feedback and performance, and what the benefits are of having multiple receive antennas. In this work, the random beamforming scheme suggested in [1] is considered. The random beamforming scheme has the transmitter with M transmit antennas produce M random orthonormal beams and send the messages on these beams. Because the throughput is a function of signal-to-interference-plus-noise ratio (SINR), if each user sends back the SINR it experiences, then the transmitter transmits to the users that are currently experiencing the best channels for each beam. If each of the n users has N receive antennas, each receive antenna of each user can be considered as an individual user and the SINR values are fed back as such. In this case, the SINR measured at each antenna for each beam results in MN SINR values to be fed back per user. Although MN SINR values are measured at each user, if the maximum per beam is sent back, namely M values, the same performance can be achieved with reduced feedback.

Other novel techniques that utilize random beamforming have since been proposed. In [2], random weight vectors are used during a training period where the users feed back the required information so that the transmitter can choose the optimal random weight vectors and users for the current transmission. This concept is extended in [3] where training is done on a set of random orthonormal bases and based upon the feedback, the transmitter will select the best beamforming vectors and users. The issue of how many random beams to use based upon the available multiuser diversity is addressed in [4]. While each of these techniques extend the methodology of random beamforming, the focus of this paper is to examine random beamforming schemes with minimal feedback. As such, only a single random orthonormal basis will be considered per fading block.

In this paper, a feedback scheme is proposed where each user sends back the maximum SINR experienced across all the receive antennas and across all the transmitted beams. This reduces the number of SINR values fed back to the transmitter further from M to 1 per user. Even with this significant reduction in feedback, it is shown that asymptotically in the number of users, the distribution of the maximum SINR is of type 3 (see [5] and [6]), the same type as that in [1]. Ultimately, using the methodology developed in [1], it is shown this reduced feedback scheme also exhibits the same asymptotic scaling as the original feedback scheme.

Next we consider an enhancement to the above schemes. In the above mentioned feedback schemes, the receive antennas are making individual SINR measurements. The availability of multiple receive antennas allows combining to be performed at

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the receiver to increase the received SINR. Transmit random beamforming is utilized in [7] with each user performing receive beamforming using the left singular vectors of the channel matrix. The effective SNR of each user is fed back to the transmitter which then selects one user to transmit to based on a proportional fair scheduler while using waterfilling for power control. This scheme will not be pursued as it is well known ([8], for example) that the optimal linear receiver is the linear MMSE receiver and using a random orthonormal basis allows parallel transmission to multiple users. Feeding back the M SINR values per user after optimal combining increases the throughput of the system for the same amount of feedback. The amount of feedback is further reduced when only the maximum SINR seen across beams after optimal combining is sent back, i.e., the maximum of those M values. This reduces the amount of feedback to a single SINR value per user and the associated beam index. The behavior of such a scheme is considered and evaluated in the paper. The distribution of the SINR values after optimal combining is derived and asymptotically the distribution of the maximum is shown to be of type 3 and to have optimal throughput scaling. Similar analysis of optimal combining in an interference limited regime is performed in [9]. This work was then extended in [10] to the general SINR setting, where for the specific case of $M = 4$ and $N = 2$, they derive the SINR distribution and scaling laws for the LMMSE receiver.

As the number of users in the system grows, the amount of feedback also grows. In an effort to further reduce the amount of feedback, a thresholding mechanism can be employed in conjunction with the previously proposed schemes. If the maximum SINR value that was to be sent back to the receiver is under the threshold, then the SINR value is not sent back. Consequently, not every user provides feedback. This scheme is also considered and it is shown that for any finite thresholding value, the loss due to thresholding asymptotically goes to zero in the number of users.

This paper extends the work in [11] in several ways, most notably by considering optimal combining, thresholding and finding a closed form expression for the distribution of the maximum SINR observed at a user when SINR measurements are taken at the antenna level.

The organization of the paper is as follows: the system model used to analyze the problem and previous work conducted in [1] is discussed in Section II. Section III considers a new scheme where only the maximum SINR per user is fed back. In the analysis of the problem, the distribution of the maximum SINR per receive antenna is derived. The asymptotic performance is also analyzed with the help of the Delta Method. Section IV extends the results of Section III to the case where each user implements an LMMSE receiver. Section V investigates the effects of thresholding the SINR feedback on the total amount of feedback in the system as well as on system throughput. Section VI summarizes the results of this paper.

II. System Model and Background

In this section, the system model used to analyze the problem and previous work found in [1] are discussed. The results found in [1] will be built upon in the latter parts of this paper.

A. System Model

A block fading channel model is assumed for the Gaussian broadcast channel. The transmitter has M transmit antennas and there are n receivers, each with N receive antennas. It is further assumed that $n \gg M$ and $N \leq M$, a reasonable assumption, for example, in a cellular system. Let $\mathbf{s}(t) \in \mathbb{C}^{M\times 1}$ be the transmitted vector of symbols at time slot t and $\mathbf{y}_i(t) \in \mathbb{C}^{N\times 1}$ be the received symbols by the i^{th} user at time slot t. The following model is used for the input-output relationship between the transmitter and the i^{th} user:

$$\mathbf{y}_i(t) = \sqrt{\rho_i}H_i\mathbf{s}(t) + \mathbf{w}_i(t), \quad i = 1, \ldots, n. \tag{1}$$

$H_i \in \mathbb{C}^{N\times M}$ is the complex channel matrix which is assumed to be known at the receiver, $\mathbf{w}_i \in \mathbb{C}^{N\times 1}$ is the white additive noise, and the elements of H_i and $\mathbf{w}_i$ are i.i.d. complex Gaussians with zero mean and unit variance (as defined by Edelman in [12]). The transmit power is chosen to be M, i.e., $E\{\mathbf{s}^*\mathbf{s}\} = M$, the SNR at the receiver is $E\{\rho_i|H_i\mathbf{s}|^2\} = M\rho_i$ and ρ_i is the SNR of the i^{th} user. It is assumed that $\rho_i = \rho\ \forall i$.

B. Background

The random beamforming scheme developed in [1] forms the basis of this work. The key elements of this work are now described.

1) SINR Distribution: The transmission scheme, as developed in [1], involves generating M random orthonormal vectors ϕ_m $(M \times 1)$ for $m = 1, \ldots, M$, where the basis is drawn from an isotropic distribution. Let $s_m(t)$ be the m^{th} transmit symbol at time t, then the total transmit signal at time slot t is given by

$$\mathbf{s}(t) = \sum_{m=1}^{M} \phi_m(t)s_m(t). \tag{2}$$

The received signal at the i^{th} user is given by

$$\mathbf{y}_i(t) = \sum_{m=1}^{M} \sqrt{\rho}H_i(t)\phi_m(t)s_m(t) + \mathbf{w}_i(t). \tag{3}$$

Each receive antenna at each user is assumed to measure the SINR for each of the M transmitted beams and the maximum of the observed SINR values is fed back leading to the use of order statistics.

Assuming that the i^{th} user knows the quantity $H_i(t)\phi_m(t)$ from (3) for all m, the SINR of the j^{th} receive antenna of the i^{th} user for the m^{th} transmit beam is computed by the following equation:

$$\text{SINR}_{i,j,m} = \frac{|H_{i,j}(t)\phi_m(t)|^2}{\frac{2}{\rho} + \sum_{k\neq m} |H_{i,j}(t)\phi_k(t)|^2}. \tag{4}$$

$H_{i,j}$ is the j^{th} row of the i^{th} user's channel matrix. Because the beamforming vectors are orthonormal and the entries of H_i are i.i.d. complex Gaussian with zero mean and unit variance, the numerator in (4) is distributed as a $\chi^2(2)$ random variable and the denominator as an independent $\chi^2(2(M-1))$ random variable. The density of the SINR is given in [1]

$$f_s(x) = \frac{e^{-\frac{x}{\rho}}}{(1+x)^M}\left(\frac{1}{\rho}(1+x)+M-1\right)u(x). \quad (5)$$

From now on, for notational simplicity, the $u(x)$ will dropped from the distribution and density expressions with the understanding that all the random variables of interest are nonnegative. To find the distribution of the maximum SINR, the distribution function must be known and is given by the integration of the density in (5) and is shown in [1] to be

$$F_s(x) = 1 - \frac{e^{-\frac{x}{\rho}}}{(1+x)^{M-1}}. \quad (6)$$

2) Scheduling Scheme: With the cdf of the SINR observed at each receive antenna known, one naive feedback scheme is for each antenna to send back all the SINR values measured for each transmit beam. This results in a total of nNM SINR values sent back to the transmitter, which then transmits to the antennas with the largest SINR for each transmit beam. Viewing each antenna as a separate user, the following approximation is shown in [1]:

$$\begin{aligned} R &\approx E\left\{\sum_{m=1}^{M}\log(1+\max_{i=1,\ldots,nN}\text{SINR}_{i,m})\right\} \\ &= ME\left\{\log(1+\max_{i=1,\ldots,nN}\text{SINR}_{i,m})\right\}. \end{aligned} \quad (7)$$

The approximation sign is required because there is a small probability that the same antenna is the best for multiple transmit beams and each beam can only be used to serve one user. To find the approximate rate requires the use of order statistics.

The distribution of the maximum SINR for a given beam is given by classical order statistics [5], [6] and is

$$F_{\max}(x) = [F_s(x)]^{nN} \quad (8)$$

where $F_s(x)$ is the CDF of the SINR given in (6). This equation can be used because the SINR values across antennas for a specific transmit beam are independent and marginally are identically distributed. Feeding back only the largest SINR value for each transmit beam per user reduces the system feedback to nM SINR values and is considered in [1]. Each SINR value fed back to the transmitter is distributed according to $[F_s(x)]^N$ since the maximum was taken over the N receive antennas prior to feedback. The transmitter takes the maximum over the n received SINR values for each beam resulting in maximum order statistic distributed according to $[F_s(x)]^{nN}$, which is identical to the distribution had every measured SINR value been fed back. Therefore, there is no benefit to feeding back all observed SINR values. For future reference, let us refer to this reduced feedback scheme of [1] as **Scheme A**.

III. Feeding Back the Largest SINR per User

Each user in **Scheme A** is feeding back the M SINR values associated with the largest SINR measured for each beam. To further reduce the amount of feedback, what happens if each user only feeds back the largest SINR value observed over all receive antennas and transmit beams as well as the associated beam index? For example, in Fig. 1, when there are four transmit beams and 3 receive antennas per user, the quantity of interest is $\max_{i,j}\text{SINR}_{i,j}$, the maximum SINR element in the grid. This scheme will reduce the total amount of system feedback from nM SINR values to n SINR values and the corresponding beam indices. This feedback scheme is referred to as **Scheme B**. Case 2 of Section VI in [1] proposes a method where at most one beam is assigned to each user and the SINR metric for user i is given by

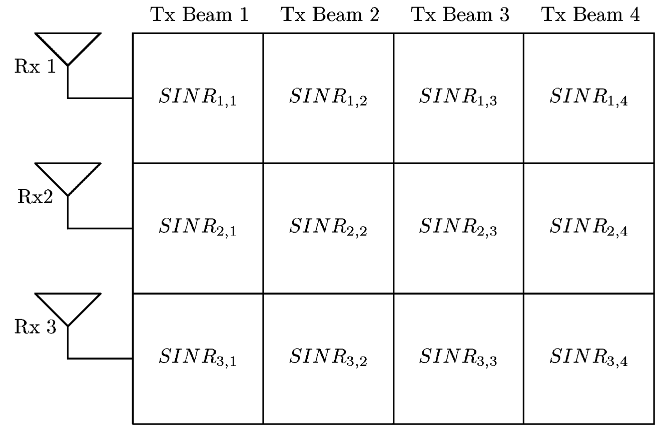

Fig. 1. SINR observations at a single user for $M = 4, N = 3$.

$$\text{SINR}_{i,j} = \frac{\phi_j^* H_i^* H_i \phi_j}{\frac{1}{\rho} + \sum_{k\neq j}\phi_k^* H_i^* H_i \phi_k}$$

which is the combined energy of transmit beam j over the sum of the inverse of the SNR and the sum of the energy from the other transmit beams. In **Scheme B**, the SINR is viewed at the antenna level for each beam while in Case 2 of [1], combining has been performed such that there is one SINR value per transmit beam. The analysis of optimal combining schemes will come in Section IV of this paper, while the current focus is on **Scheme B**. The analysis of this scheme poses some difficulties that do not arise when considering **Scheme A**. This section addresses the new difficulties that arise and then, in the same vein as the work of [1], the asymptotic performance of **Scheme B** is analyzed.

A. Analysis of Scheme B

Although **Scheme B** reduces the amount of feedback by a factor of M compared to **Scheme A**, it is suboptimal. This is due to the fact that the best SINR values for a particular beam may not be fed back due to the restriction that only one value can be sent back per user. This restriction, however, removes the mechanism which led to the approximation symbol in (7). Although **Scheme B** may be suboptimal, in the interest of reducing the feedback as much as possible, this scheme is considered. Later in this section the asymptotic performance of the scheme in the number of users will be analyzed and will be shown to have optimal scaling properties.

The distribution of the SINR that is served by the transmitter is fundamentally different for **Scheme B** than **Scheme A** for two reasons. The first difference is that in **Scheme A**, the maximum is taken for each beam and the marginal distributions of the SINR were i.i.d. across receive antennas. However, the SINR values at a particular receive antenna for different transmit beams are coupled. Looking again at Fig. 1, the SINR values in a given column are i.i.d., where as the SINR values in a given row are marginally identically distributed but are not independent. To see this, fix the antenna at a particular user and vary the beam index. Let $|H_{i,j}(t)\phi_m(t)|^2 = X_m$ and $c = 2/\rho$. Then the SINR values at a receive antenna are given by $(X_1/(c+\sum_{k\neq 1} X_k)), \ldots, (X_M/(c+\sum_{k\neq M} X_k))$. The SINRs are coupled by the appearance of the numerator term of a particular SINR value appearing in the denominator of all the other SINR values. Because the SINR values are not independent, the order statistics used earlier cannot be applied. The second difference between the two schemes is that in **Scheme B** since each user feeds back the largest observed SINR over all receive antennas and transmit beams, the number of SINR values to maximize over at the transmitter for each beam is a random quantity. In terms of the distribution of the maximum order statistics, this changes the exponent reflecting the number of variables being maximized over. These fundamental differences will now be addressed.

1) Distribution of the Maximum SINR per User: If the distribution of the largest SINR value at a particular user for a fixed receive antenna can be found, then the maximum SINR per user can be found because the SINR random variables are independent across receive antennas. Using the notation defined earlier, let $|H_{i,j}(t)\phi_m(t)|^2 = X_m$ and $c = 2/\rho$. Then for a fixed receive antenna, the SINR for each transmit beam is given by

$$\text{SINR}_1 = \frac{X_1}{c+\sum_{i\neq 1}^{M} X_i}, \ldots, \text{SINR}_M = \frac{X_M}{c+\sum_{i\neq M}^{M} X_i}.$$

The distribution of interest is the maximum of these M SINR values. The key observation is that if the X_is are ordered, i.e., $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(M)}$, then the maximum SINR for a fixed receive antenna is given by

$$\text{SINR}_{(M)} = \frac{X_{(M)}}{c+\sum_{i=1}^{M-1} X_{(i)}}. \quad (9)$$

This makes intuitive sense, since to maximize the SINR, the largest signal power should be put in the numerator and all the other signal powers should be considered as interference and put in the denominator. To find the distribution of the quantity in (9), results based on the ratio of the linear combination of order statistics are called upon. As mentioned previously, the X_is are distributed as a χ_2^2 random variable, which is equivalent to a exponential-(1/2) random variable. The ratio of the linear combination of order statistics drawn from an exponential distribution have been studied in [5] and [13]. After some manipulation to get the χ_2^2 random variables in the proper exponential form, the distribution function of the maximum SINR for a particular beam index is given by

$$\Pr\left(\text{SINR}_{(M)} \leq x\right) = F_{\text{SINR}_{(M)}}(x) = -\sum_{i=1}^{M} \frac{[d_i(x)]_+^M \exp\left(-\frac{xc}{d_i(x)}\right)}{A_i(x)} \quad (10)$$

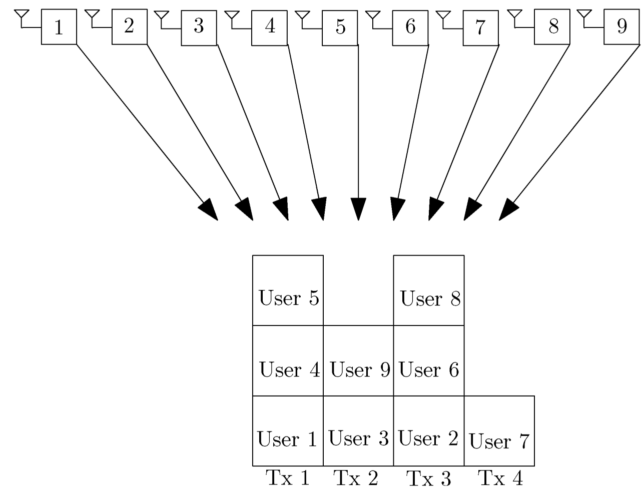

Fig. 2. 9 user system only feeding back beam index and largest SINR over transmit beams and receive antennas.

where $d_i(x) = 2[1 - x(M-i)]/(M-i+1)$, $A_i(x) = d_i(x)\prod_{j\neq i}^{M}(d_i(x) - d_j(x))$, and $[\cdot]_+$ is the positive part of the argument.

Equation (10) gives the distribution of the maximum SINR over the beams for a particular receive antenna at a specific user. **Scheme B** feeds back the largest SINR per user, so the maximum has to be taken over receive antennas as well. As with **Scheme A**, the random variables across receive antennas are independent, thus the distribution of the maximum SINR per user is given by $[F_{\text{SINR}_{(M)}}(x)]^N$.

2) Order Statistics Over a Random Number of Observations: If each user feeds back the maximum observed SINR and the beam index that produced it, then the transmitter receives n SINR values and n beam indexes over which to maximize. The number of SINR values to maximize over at the transmitter for a particular beam is, however, a random number. For example, in Fig. 2, there are nine users and at the transmitter, the number of users feeding back a particular beam index is random. In this case three users feed back beam index 1, two users feed back beam index 2, and so on. This effect is not taken into consideration in Case 2, Section VI of [1]. There it is mentioned that only the maximum SINR (which differs from the SINR metric currently under consideration) and the corresponding beam index need be fed back, yet the maximum is taken over all n users even though no information is known about many of the users for a particular beam since that information will not be fed back.

Because the beamforming vectors are a randomly generated orthonormal basis from an isotropic distribution and the matrices are composed of i.i.d. circularly symmetric complex Gaussian random variables, there is no preferred beam over time. That is, each beam has an equal probability of being the one that produced the maximum at any given user. The joint distribution of the number of SINR values fed back for each beam can then be viewed as a multinomial distribution where the probability of each beam being selected is $1/M$. The distribution of the order statistic of the maximum SINR is a distribution function raised to a power that is a random variable. Marginally, the selection of each beam is distributed binomially with probability $1/M$. Averaging over the binomially

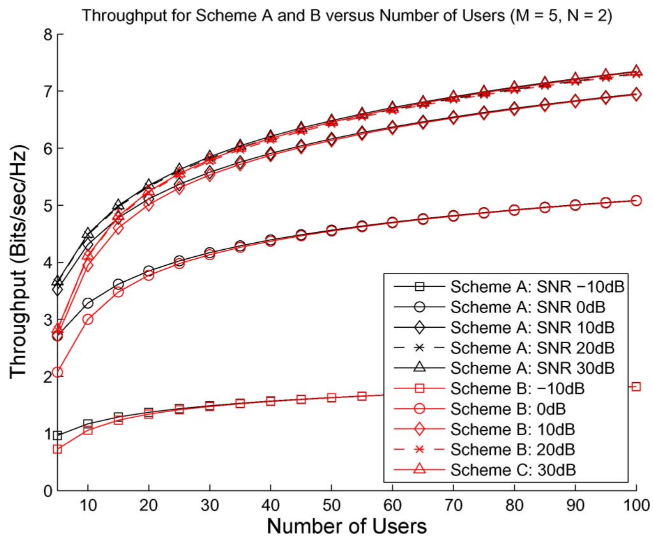

Fig. 3. Throughput as a function of number of users for different schemes for $M = 5$, $N = 2$ and various SNRs.

distributed exponent applied to the distribution $F_{\mathrm{SINR}_{(M)}}(x)$ given by (10) yields

$$F_{SchemeB}(x) = \sum_{i=0}^{n} \left[F_{\mathrm{SINR}_{(M)}}(x)\right]^{Ni} \binom{n}{i} \times \left(\frac{1}{M}\right)^{i} \left(1 - \frac{1}{M}\right)^{n-i}. \quad (11)$$

3) Throughput Analysis: Using integration by parts and (7), the throughput of a scheme is expressed as

$$R = ME\{\log(1+X)\} = M \int_{0}^{\infty} \frac{1}{1+x}(1 - F(x))\, dx \quad (12)$$

where X is drawn from the distribution of the scheme being used. Numerical integration is very attractive since the closed form expression of the expectations in (6) for the distributions derived earlier are not known to the authors. Fig. 3 compares the throughput of **Scheme A** with **Scheme B** as a function of the number of users for various SNRs. Notice that **Scheme A** and **Scheme B** tend towards each other as the number of users increases. This seems reasonable since as the number of users in the system increases, it is expected that the maximum SINR for each beam is distributed across users with high probability. When the maximum SINR for each beam is distributed over users, **Scheme B** captures the true maximum and the two schemes perform equivalently. Fig. 3 also shows that as the SNR increases, eventually there is no performance gain due to the increased interference. Most noticeably, at SNRs of 20 and 30 dB, the performance is virtually indistinguishable.

B. Asymptotic Performance

The asymptotic performance of these schemes compared with the optimal sum-rate capacity that is achieved via dirty paper coding is of primary interest. The main result from [1] concerns the asymptotic scaling of **Scheme A** with one receive antenna per user and is restated here.

Theorem 1 [1]): Let M and ρ be fixed and $N = 1$. Then

$$\lim_{n\to\infty} \frac{R}{M \log\log n} = 1 \quad (13)$$

where R is the throughput of **Scheme A**.

In the single receive antenna case, i.e., $N = 1$, the value of $M \log\log n$ comes from the fact that the transmitter is selecting the maximum SINR from n values for each beam. For a single receive antenna per user, it is known that the optimal sum-rate capacity scales as $M \log\log n$, and so the scaling is optimal. For **Scheme A** in the more general case where $1 < N \leq M$, the sequence of random variables for each transmit beam seen at the transmitter is $x_1, \ldots, x_{nN}$, thus the dominator in Theorem 1 becomes $M \log\log nN$, as pointed out in Section VI, Case 1 of [1]. The following corollary summarizes the more general case.

Corollary 1 [1]: Let M, $N \leq M$, and ρ be fixed. Then for **Scheme A**, the throughput satisfies

$$\lim_{n\to\infty} \frac{R}{M \log\log nN} = 1.$$

It should briefly be noted that a scaling rate of $M \log\log nN$ is essentially the same rate as $M \log\log n$ since N is a constant that inside the double logarithm becomes inconsequential in the limit as n goes to infinity, i.e., $\lim_{n\to\infty}(\log\log nN - \log\log n) = 0$.

The denominator term in Theorem 1 is determined by the number of observations that the maximum is taken over. **Scheme B** feeds back only the largest SINR measured at each user at the antenna level. The sequence of random variables to be maximized over for each beam is then $x_1, \ldots, x_{M_i}$, where M_i is the number of SINR values fed back for the i^{th} beam. This led to the binomial type expression of the maximum SINR. What is the scaling when the number of terms to be maximized over is random?

Theorem 1 is concerned with the asymptotic scaling relative to the sum-rate capacity. Using the strong law of large numbers, as the number of users n grows, the total number of values fed back for each transmit beam converges to n/M. In **Scheme B** let $Y_{i,n}$ be the sequence of random variables in n denoting the number of SINR values fed back for the i^{th} beam in a system with n users. Let $\theta = 1/M$ be the probability that a particular transmit beam is selected. The sequence of random variables $Y_{i,n}$ are binomially distributed with probability of success θ. Define $X_{i,n} = Y_{i,n}/n$. Then this sequence of random variables, by the central limit theorem, satisfies

$$\lim_{n\to\infty} \sqrt{n}\left(X_{i,n} - \frac{1}{M}\right) \Rightarrow \mathcal{N}\left(0, \frac{1}{M}\left(1 - \frac{1}{M}\right)\right). \quad (14)$$

Let $\sigma^2 = (1/M)(1-(1/M))$ and define the function $g(x,\theta) = [F_{\mathrm{SINR}_{(M)}}(x)]^{nN\theta}$. The motivation for these equations is that as the number of users in the system increases, with high probability the number of SINR values fed back per beam approaches

$n\theta$. The equation $g(x,\theta)$ is selected from the term being binomially averaged in (11). The notion of each beam being the maximum for approximately $n\theta$ users with high probability is made rigorous by the Delta Method ([16])

$$\sqrt{n}\left[g(x,X_{i,n})-g(x,\theta)\right]\Rightarrow\mathcal{N}\left(0,\sigma^2\left[\frac{d}{d\theta}g(x,\theta)\right]^2\right). \quad (15)$$

Plugging the values into this equation yields

$$\left[\left(F_{\mathrm{SINR})_{(M)}}(x)\right)^{nNX_{i,n}}-\left(F_{\mathrm{SINR}_{(M)}}(x)\right)^{n\frac{N}{M}}\right]\Rightarrow \mathcal{N}\left(0,n^2M\left(1-\frac{1}{M}\right)\left[F_{\mathrm{SINR}_{(M)}}(x)\right]^{2\frac{nN}{M}} \times\left(\log\left(F_{\mathrm{SINR}_{(M)}}(x)\right)\right)^2\right) \quad (16)$$

for a particular argument x. Because the distribution $F_{\mathrm{SINR}_{(M)}}(x) < 1$ for any finite argument, the variance of the above distribution tends to zero as the number of users in the system increases. Therefore, estimating any point of the distribution of the extreme order statistic of the SINR where only the maximum SINR seen at each user is fed back can be approximated by $[F_{\mathrm{SINR}_{(M)}}(x)]^{nN/M}$. This yields the following corollary.

Corollary 2: The throughput of **Scheme B** for fixed $M, N \leq M$, and ρ scales as $M\log\log(nN/M)$, or

$$\lim_{n\to\infty}\frac{R}{M\log\log\frac{nN}{M}}=1.$$

Before proving the corollary, note that the effect of having the number of users grow to infinity is to provide more observation to take the maximum over. In **Scheme B** where the feedback is restricted to solely one SINR value, asymptotically in the number of users, the performance scales doubly logarithmically with nN/M rather than nN as in **Scheme A**. As mentioned before, the multiplicative factor of $1/M$ difference between the schemes is inconsequential in the limit since the growth rate is double logarithmic.

Proof: It is well known [5], [6] that if there exists a limiting distribution of the maximum order statistic, the limiting distribution is one of three types. It will be shown that the distribution of the asymptotic order statistic of $F_{\mathrm{SINR}_{(M)}}(x)$ in the terminology of [5], is of type 3, i.e.,

$$\Lambda_3(x)=e^{-e^{-x}} \quad -\infty<x<\infty. \quad (17)$$

A well-known condition for the asymptotic distribution to be of type 3 is

$$\lim_{x\to\infty}\frac{d}{dx}\left[\frac{1-F(x)}{f(x)}\right]=0. \quad (18)$$

Carrying out the differentiation, another equivalent condition for the asymptotic distribution to be of type three is the following:

$$\lim_{x\to\infty}\frac{[F(x)-1]\,f'(x)}{(f(x))^2}=1 \quad (19)$$

where $f(x)$ is the density of $F(x)$, which exists since our distribution is continuous.

To show that $F_{\mathrm{SINR}_{(M)}}$ is of type 3, it is shown that it satisfies (19). Since differentiation is a local property of a function and (19) is concerned with the limit as the argument goes to infinity, significant simplification of the distribution $F_{\mathrm{SINR}_{(M)}}$ can be made because the terms $[d_i(x)]_+^M=0$ for $x\geq 1$ except when $i=M$. Therefore, the distribution simplifies to

$$F_{\mathrm{SINR}_{(M)}}(x)=1-\frac{e^{-\frac{x}{\rho}}}{\alpha(x+1)^{M-1}} \text{ for } x\geq 1 \quad (20)$$

where $\alpha=\prod_{j=1}^{M-1}((j-M)/(j-M-1))$. Equation (19) can be verified by taking the first and second derivatives of (20), and thus $F_{\mathrm{SINR}_{(M)}}$ is of type 3.

Notice that (20) is equivalent to (6) except for the α term. Following the analysis of [1], to satisfy the conditions of Uzgoren's theorem [15], it must be shown that there exists a $u_n=O(\log n)$ such that

$$1-F_{\mathrm{SINR}_{(M)}}(u_n)=\frac{e^{-\frac{x}{\rho}}}{\alpha(x+1)^{M-1}}=\frac{1}{n}. \quad (21)$$

For sufficiently large n such that (20) holds, the existence of a unique u_n satisfying (21) is guaranteed since $e^{-(x/\rho)}/\alpha(x+1)^{M-1}$ is continuous and monotonically decreasing. To show that $u_n = O(\log n)$, notice that rearranging (21) yields $(u_n/\rho)+(M-1)\log(1+u_n) = \log n-(M-1)\log\alpha$, where $(M-1)\log\alpha$ is constant for fixed M. Thus, the same conclusion as [1] is reached, that $u_n=\rho\log n-\rho(M-1)\log\log n+O(\log\log\log n)$, where the constant term can be absorbed by the O-notation. With the above results, the fact $F_{\mathrm{SINR}_{(M)}}$ is of type 3, and the results from Appendix A, Theorem 1 holds exactly as in [1], except for one key difference. In [1], the rate scales with the number of users n, but from the above analysis, for a particular beam, the number of values being fed back is converging to nN/M, which yields Corollary 2. ■

IV. MMSE Receivers and Feedback

The previous feedback schemes measured the SINR at the individual antenna level. However, since each user in the system has N receive antennas, a more complex receiver structure can be utilized. It is known that the optimal linear receiver is the linear MMSE receiver. Using the system model defined in (1)–(3), the SINR after optimal combining for the i^{th} user and the j^{th} transmit beam is given by

$$\mathrm{SINR}_{i,j}=\phi_j^*H_i^*\left[H_i\left(\sum_{k\neq j}\phi_k\phi_k^*\right)H_i^*+\frac{2}{\rho}I\right]^{-1}H_i\phi_j \quad (22)$$

where * denotes conjugate transposition and I is the identity matrix.

Once all the SINR values after optimal combining are computed for all the transmit beams at each receiver, the transmitter requires feedback for user selection. Prior to any maximization, there are M SINR values after optimal combining at each user. Immediately one could implement a scheme, call it **Scheme C**, similar to **Scheme A** and feed back the SINR values for each transmit beam after linear MMSE combining. Alternatively, if M SINR values represent too much data to feedback to the transmitter, a scheme similar to **Scheme B** can be adopted, call it **Scheme D**, where the maximum SINR after optimal combining can be sent back. In **Scheme D**, the total amount of feedback in the system is reduced to n analog SINR values and the corresponding n integer beam indices. To analyze **Scheme C** and **Scheme D** requires the analysis of the distribution of the SINR after optimal combining given by (22), which leads to the following theorem.

Theorem 2: The distribution function of the post-processing SINR given by (22) for the system defined in (1)–(3) is given by

$$F_{\mathrm{MMSE}}(x) = 1 - \frac{\exp\left(-\frac{x}{\rho}\right)}{(1+x)^{M-1}} \times \sum_{i=1}^{N} \frac{1+\sum_{j=1}^{N-i}\binom{M-1}{j}x^j}{(i-1)!}\left(\frac{x}{\rho}\right)^{i-1}. \quad (23)$$

Proof: See Appendix B. ■

As mentioned in the introduction, the distribution function for the interference limited regime is analyzed in [9], while the SINR distribution is derived for the case $M=4$ and $N=2$ in [10].

The analysis of **Scheme C** is straight forward because at the transmitter the values fed back from each user for each beam are independent and identically distributed according to the distribution given by (23). Therefore, the results from classical order statistics apply, namely the distribution of the maximum SINR selected by the transmitter for each beam is given by $[F_{\mathrm{MMSE}}(x)]^n$.

The analysis of **Scheme D** becomes difficult in the same fashion that the analysis of **Scheme B** becomes difficult, namely the SINR values after optimal linear combining are correlated. Unlike the situation in **Scheme B** where the distribution of the maximum order statistic of the correlated random variables could be found, the distribution of the maximum order statistic of the SINR values after optimal combining at a particular user could not be found. Since the explicit distribution could not be found, bounds on the true distribution function will be used. The bounds of interest are the classical Fréchet bounds.

Theorem 3 [6], The Fréchet Bounds: Let $F(\mathbf{X})$ be an m-dimensional distribution function with marginals $F_j(x)$, $1 \le j \le m$. Then, for all $x_1, x_2, \ldots, x_m$

$$\max\left(0, \sum_{j=1}^{m} F_j(x_j) - m + 1\right) \le F(x_1, x_2, \ldots, x_m) \le \min(F_1(x_1), \ldots, F_m(x_m)).$$

The Fréchet bounds provide control of the unknown joint distribution of the SINR values after optimal combining, and can provide bounds on the order statistic because all the marginal distributions are identical:

$$\max\left(0, M\left(F_{\mathrm{MMSE}}(x) - 1\right) + 1\right) \le F_{\mathrm{MaxMMSE}}(x) \le F_{\mathrm{MMSE}}(x) \quad (24)$$

where F_{MaxMMSE} is the distribution of the maximum SINR at a user after optimal combining.

The asymptotic performance of **Scheme C** and **Scheme D** are of interest. To show that **Scheme C** has optimal scaling asymptotically, it must first be shown that $F_{\mathrm{MMSE}}(x)$ given by (23) is of type 3 (17).

Theorem 4: The limiting distribution of the extreme order statistics drawn from the distribution $F_{\mathrm{MMSE}}(x)$ is of type 3.

Proof: See Appendix C. ■

Having established the previous theorem, the scaling rate is shown in Appendix C to satisfying the following corollary:

Corollary 3: For fixed M, $N \le M$, and ρ, the throughput for **Scheme C** asymptotically scales as $M \log\log n$, or

$$\lim_{n\to\infty} \frac{R}{M\log\log n} = 1.$$

Compared with **Scheme A**, the asymptotic scaling is $\log\log n$ rather than $\log\log nN$, but in the limit the extra factor of N becomes insignificant. The loss of the factor of N comes from the fact that in **Scheme C** only one SINR value (after optimal combining) is fed back per user and in **Scheme A** the SINR value that was the maximum of N SINR values (without optimal combining) was fed back, resulting in the extra factor of N.

Theorem 4 establishes that $F_{\mathrm{MMSE}}(x)$ is of type 3. The bounds in (24) are in terms of $F_{\mathrm{MMSE}}(x)$ and thus are also of type 3 implying the true unknown distribution of the maximum order statistic is of type 3. As with **Scheme B**, the number of SINR values after optimal combining that are maximized over at the transmitter in **Scheme D** is a random quantity, leading to binomial averaging. The binomial averaging can be applied to both the bounds given by (24) and the Delta Method arguments will yield an exponent in both bounds asymptotically approaching n/M. To get a handle on the asymptotic scaling, consider the lower and upper bounds in (24). The upper bound scales as $M\log\log n$ as shown in Appendix C. Substituting the distribution into the lower bound yields $\max\{0, 1 - MR(z)\}$, where $R(z)$ is defined by (34) in Appendix B. For sufficiently large n, $\max\{0, 1-MR(z)\} = 1 - MR(z)$ since $R(z)$ is monotonically decreasing to zero, so by the same methods as Appendix C, the lower bound scales as $M\log\log n$. Both the lower and upper bounds have the same asymptotic scaling rate, yielding the following corollary.

Corollary 4: For fixed M, $N \le M$, and ρ, the throughput for **Scheme D** asymptotically scales as $M\log\log n$, or

$$\lim_{n\to\infty} \frac{R}{M\log\log n} = 1.$$

The upper bound in (24) is very loose. Although it could not be shown, empirical evidence suggests that the SINR values after optimal combining are negatively associated. The definition of negative association is as follows:

Definition 1 [18], Negative Association: Random variables $X_1, X_2, \ldots, X_k$ are said to be *negatively associated* if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, k\}$,

$$\mathrm{Cov}\left\{f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)\right\} \le 0$$

whenever f_1 and f_2 are both increasing or both decreasing.

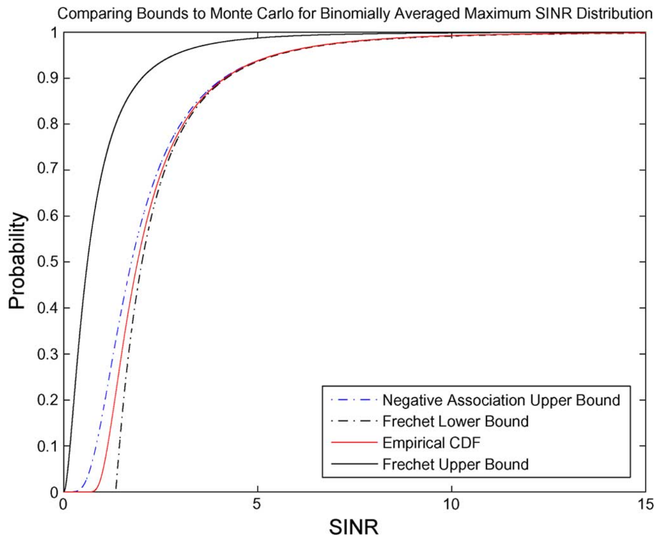

Fig. 4. Theoretical versus empirical distribution for SINR after optimal combining for $M = 5, N = 2$, SNR $= 10$ dB.

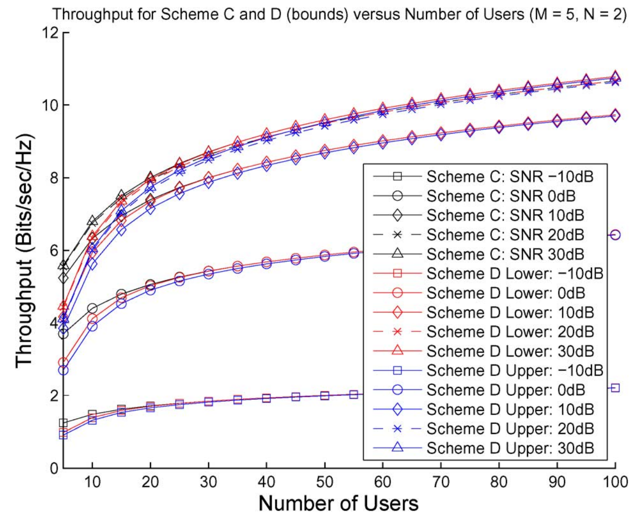

Fig. 5. Throughput as a function of number of users for different schemes for $M = 5, N = 2$ and various SNRs.

Negative association is a multivariate generalization of negative correlation. The key property of interest is that if $X_1, \ldots, X_N$ are negatively associated random variables, then

$$\Pr(X_1 \leq x_1, \ldots, X_N \leq x_N) \leq \prod_{i=1}^{N} \Pr(X_i \leq x_i).$$

Applying this to our distribution, we conjecture a tighter upper bound is given by

$$F_{\mathrm{MaxMMSE}}(x) \leq [F_{\mathrm{MMSE}}(x)]^M. \tag{25}$$

Fig. 4 shows the bounds versus Monte Carlo simulations. Although the original upper bound due to Fréchet is weak, it sufficed in showing the optimal asymptotic scaling since the bound was of type 3. The tighter hypothesized bound due to negative association may be of use in numerical computations.

Fig. 5 shows **Scheme C** and the upper and lower bounds for **Scheme D**. The upper bound is the conjectured bound due to negative association. The figure shows that the bounds on **Scheme D** converge as the number of users increases and that they converge to the rate of **Scheme C**. As with **Scheme A** compared with **Scheme B**, this convergence is expected as the maximum for each transmit beam will be distributed over users with high probability as the number of users increases. The throughput also becomes saturated as the SNR increases and the interference becomes dominant. Comparing Fig. 3 and Fig. 5, utilizing the extra receive antenna for optimal combining rather than just additional observations yields significant throughput gains at the expense of receive complexity.

V. Thresholding the Feedback

Feedback is a precious commodity and the amount of feedback may have to be reduced to a bare minimum. One method of further reducing the amount of feedback is to apply a threshold at the receivers, and only send back the SINR values that exceed the threshold. Intuitively, if the SINR is below a reasonable threshold, it is very unlikely to be the maximum selected by the transmitter. This idea is briefly mentioned in [1] for **Scheme A**, where it is mentioned that as n grows large, the SINR value need only be fed back if it exceeds some constant threshold η, which is independent of n, to maintain the scaling laws. The rate lost in a system with a finite number of users is not considered. The analysis in this section determines the effects on throughput of applying any fixed threshold to a system with an arbitrary number of users, and then applying that analysis to show asymptotically there is no loss in throughput by applying any finite threshold. Additionally, by applying an appropriate threshold, one can trade off throughput for a reduction in feedback.

It is of interest to see how thresholding the SINR affects the distributions over which the expectations are taken to determine the throughput of the previous schemes. No closed form expression for the expectation could be found due to the complicated nature of the distributions. The effect of thresholding is to truncate the underlying cdf of the SINRs observed at the users in the following manner:

$$F_{\mathrm{threshold}}(x) = \begin{cases} F(x_{\mathrm{th}}) & x \leq x_{\mathrm{th}} \\ F(x) & x > x_{\mathrm{th}} \end{cases} \tag{26}$$

where x_{th} is the threshold. All SINR values less than the threshold are not fed back, which can be viewed as a mapping to zero throughput, resulting in the above truncation.

For concreteness, consider **Scheme A**. Applying (12) yields

$$R \approx M \int_0^\infty \frac{1}{1+x} \left(1 - F_s(x)^{nN}\right) dx. \tag{27}$$

After thresholding, the throughput of the scheme is

$$R_{\text{threshold}} \approx M\left[\int_0^{x_{\text{th}}} \frac{1}{1+x}\left(1-[F_s(x_{\text{th}})]^{nN}\right)dx + \int_{x_{\text{th}}}^{\infty} \frac{1}{1+x}\left(1-F_s(x)^{nN}\right)dx\right]$$
$$= M\left[\log(1+x_{\text{th}})\left(1-[F_s(x_{\text{th}})]^{nN}\right) + \int_{x_{\text{th}}}^{\infty} \frac{1}{1+x}\left(1-F_s(x)^{nN}\right)dx\right]$$

which is a constant term plus an integral.

The loss in throughput due to thresholding can be expressed as

$$R-R_{\text{threshold}} = M\int_0^{\infty} \frac{1}{1+x}[F_{\text{threshold}}-F_{\max}]dx$$
$$= M\left[\int_0^{x_{\text{th}}} \frac{1}{1+x}\left[F_{\max}(x_{\text{th}})-F_{\max}(x)\right]dx + \int_{x_{\text{th}}}^{\infty} \frac{1}{1+x}\left[F_{\max}(x)-F_{\max}(x)\right]dx\right]$$
$$= M\left[\log(1+x_{\text{th}})F_{\max}(x_{\text{th}}) - \int_0^{x^{\text{th}}} \frac{1}{1+x}F_{\max}(x)dx\right]$$
$$\le M\log(1+x_{\text{th}})F_{\max}(x_{\text{th}})$$

where $F_{\max}$ could be any of the maximum distributions derived earlier. As the number of users grows, the upper bound on the throughput loss goes to zero for all the distributions discussed, since for a finite threshold, $F_{\max}(x_{\text{th}}) \to 0$ as the number of users increases. Therefore, asymptotically, if the threshold is finite, the scaling laws derived earlier hold.

For **Scheme A**, the number of SINR values not sent back due to thresholding is $nMF_s(x_{\text{th}})$, that is the $\Pr[\text{SINR} \le x_{\text{th}}] = F_s(x_{\text{th}})$ multiplied by the number of possible values that could be sent back. Viewing it another way, it is a binomial experiment with nM trials and the probability of success being $F_s(x_{\text{th}})$. Similarly, using the cdf of the SINR after optimal combining, the number of SINR values saved for **Scheme C** is on average $nMF_{\text{MMSE}}(x_{\text{th}})$. For **Scheme B** the amount of savings is on average $n[F_{\text{SINR}_{(M)}}(x_{\text{th}})]^N$ and the savings for **Scheme D** can be bounded using the Fréchet bounds.

Fig. 6 shows **Scheme B** without thresholding against **Scheme A** for various levels of thresholding. For small thresholds, there is almost no change in throughput performance, whereas for the higher thresholds, large losses are suffered for small numbers of users. As the number of users is increased, there is no loss in performance as expected. **Scheme B** performs better than **Scheme A** for the threshold of 1 for small to moderate number of users, but does not uniformly bound the thresholded version of **Scheme A**. It is not quite a far comparison since **Scheme B** deterministically feeds back only one SINR value and the associated beam index, while applying a threshold to **Scheme A** yields a reduction in feedback that is a random variable. In this vein, it may be of interest to design the threshold to achieve an average amount of feedback per fading period.

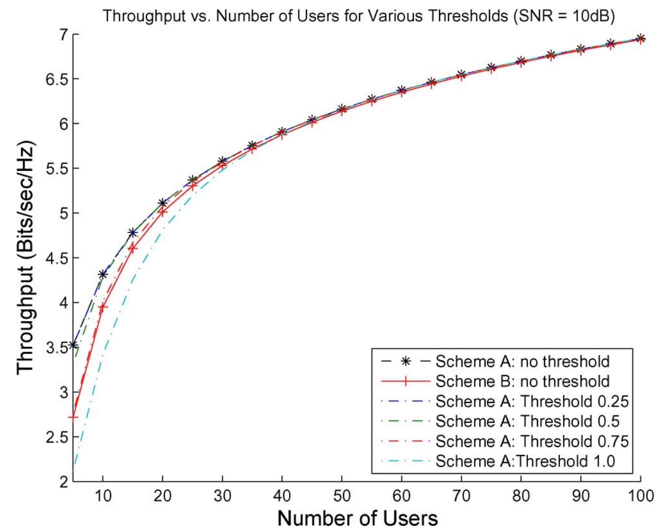

Fig. 6. Throughput as a function of number of users for various threshold levels.

VI. Conclusion

This paper is concerned with a random beamforming scheme that puts a premium on the amount of feedback in the system. To first reduce the amount of feedback, a metric was chosen that captures the quality of the channel, and that metric was the SINR, and has been motivated by its use in the formula for throughput of the channel. With a good metric chosen, a few schemes that feed back this SINR information are considered.

First, the scheme in [1] (**Scheme A**) was reviewed to provide background and context for different novel schemes. **Scheme B** suggests utilizing knowledge at the receiver and send back the maximum SINR observed at each user over all beams. This reduces the amount of feedback in the system to n SINR values. Since each user has N receive antennas, it is possible to perform optimal combining at the receivers. The distribution for the SINR after optimal combining is derived. There are two schemes for utilizing this optimally combined SINR value: either all M SINRs could be fed back per user (**Scheme C**), or send back only the maximum SINR at each user leading to system total of n feedback values (**Scheme D**). The asymptotic scaling of all the schemes is shown to be essentially $M \log\log n$.

Finally, thresholding was considered as another means of reducing the amount of feedback. If the SINR value to be sent back in any scheme lies below the set threshold, that SINR value is not sent back at all. Asymptotically in the number of users, the throughput lost due to thresholding at any finite level is shown to go to zero.

APPENDIX A

Let $x_1, \ldots, x_n$ be a sequence of positive random variables with strictly positive density function $f_X(x)$ on the positive real line and cdf $F_X(x)$. The growth function $g_X(x)$ is defined to be

$$g_X(x) = \frac{1 - F_X(x)}{f_X(x)}. \tag{28}$$

Also, define the variable u_n to be the unique solution to

$$1 - F_X(u_n) = \frac{1}{n}. \tag{29}$$

With these definitions in hand, the theorem due to Uzgoren is restated.

Theorem 5: (Uzgoren [15]): Let $x_1, \ldots, x_n$ be a sequence of i.i.d. positive random variables with continuous and strictly positive pdf $f_X(x)$ for $x > 0$ and cdf of $F_X(x)$. Let also $g_X(x)$ be the growth function. Then if $\lim_{x\to\infty} g_X(x) = c > 0$, then

$$\log\{-\log F^n(u_n + u g_X(u_n))\} = -u - \frac{u^2 g_X'(u_n)}{2!} - \cdots - \frac{u^m g_X^{(m)}(u_n)}{m!} + O\left(\frac{e^{-u+O\left(u^2 g_X'(u_n)\right)}}{n}\right).$$

The distributions for the maximum SINR per beam and the optimal combining SINR have support on the non-negative real line and have continuous cdfs that do not attain the value of unity for any finite value of the support, which implies the densities are strictly positive on the support. Thus, it must be shown that $\lim_{x\to\infty} g_X(x) = c > 0$. It is shown more generally that having an asymptotic maximum order statistic distribution of type 3 implies this condition. Equation (19) gives a condition for the asymptotic distribution of the maximum order statistic to be of type 3, and it can be rewritten as

$$\lim_{x\to\infty}\left(-1 - \left[\frac{1 - F_X(x)}{f_X(x)}\right]\left[\frac{f_X''(x)}{f_X(x)}\right]\right) = \lim_{x\to\infty}\left(-1 - g_X(x)\frac{f_X''(x)}{f_X(x)}\right) = 0. \tag{30}$$

It was shown that the distributions of interest satisfy this limit. Thus, for the limit to go to zero, we need $\lim_{x\to\infty} g_X(x)(f_X''(x)/f_X(x)) \to -1$. Because $1 - F_X(x)$ and $f_X(x)$ are non-negative for all x in the support of the distribution, $\lim_{x\to\infty} g_X(x) \geq 0$. From basic properties of limits, it is also known that

$$\lim_{x\to\infty} g_X(x)\frac{f_X''(x)}{f_X(x)} = \lim_{x\to\infty} g_X(x) \lim_{x\to\infty}\frac{f_X''(x)}{f_X(x)} = -1. \tag{31}$$

Therefore, since the limit of the product is finite and non-zero, $\lim_{x\to\infty} g_X(x) = c > 0$, and the conditions of the theorem are satisfied.

Next, it must be shown that the following corollary shown in the appendix of [1] holds.

Corollary 5: Let $x_1, \ldots, x_n$ be a sequence of i.i.d. positive random variables with continuous and strictly positive pdf $f_X(x)$ for $x > 0$ and cdf of $F_X(x)$. If $u_n = O(\log n)$, $\lim_{x\to\infty} g_X(x) = c > 0$, and $g_X^{(m)}(u_n) = O(1/u_n^m)$, then

$$\Pr\{u_n - c\log\log n \leq \max x_i \leq u_n + c\log\log n\} \geq 1 - O\left(\frac{1}{\log n}\right). \tag{32}$$

All the conditions of this corollary except for the derivative constraint were previously shown to hold. Suppose that $g_X^{(1)}(x) = \Theta(1/x)$, where $f(x) \in \Theta(h(x))$ means f is asymptotically bounded above and below by h, i.e., asymptotically $|h(x)|k_1 \leq |f(x)| \leq |h(x)|k_2$ for some k_1, k_2. Then by integration $g_X(x) = \Theta(\log x)$, but $\lim_{x\to\infty} \log x \to \infty$, which contradicts $\lim_{x\to\infty} g_X(x) \to c > 0$, therefore $g_X^{(1)}(x) = o(1/x)$. This implies that $g_X^{(m)}(x) = o(1/x^m)$, and because $o(f(x)) \subset O(f(x))$, the derivative constraint of Corollary 5 is also met.

APPENDIX B

The distribution of the SINR from (22) turns out to be a special case of the work in [20] and [21]. In [20], Gao and Smith let the random variable Z denote the SINR at the output of the optimal combiner and were interested in the link reliability

$$R(z) = \Pr[Z \geq z]. \tag{33}$$

The quantity of interest in this paper is the cdf of Z as to utilize order statistics, which is given by the quantity $F(z) = 1 - R(z)$. Define $E[H_i\phi_j\phi_j^* H_i^*] = P_i\mathbf{I}$ for the i^{th} user and j^{th} transmit beam, and the noise covariance $\sigma^2\mathbf{I}$. Assume the j^{th} beam is the intended signal, then let $\gamma = \rho P_i/\sigma^2$ and $\Gamma_k = P_k/P_j$. For the additive Gaussian noise channel where there are $M - 1$ interferers for a given beam (i.e., the other $M - 1$ beams), Equations (11)–(13) in [20] define the function $R(z)$:

$$R(z) = \exp\left(-\frac{z}{\gamma}\right)\sum_{i=1}^{N}\frac{A_i(z)}{(i-1)!}\left(\frac{z}{\gamma}\right)^{i-1} \tag{34}$$

where

$$A_i(z) = \begin{cases} 1, & N \geq M + i \\ \frac{1 + \sum_{j=1}^{N-i} C_j z^j}{\prod_{k=1}^{M-1}(1 + \Gamma_k z)}, & N < M + i. \end{cases} \tag{35}$$

The coefficient C_j in (35) is the coefficient of z^j in $\prod_{k=1}^{M-1}(1 + \Gamma_k z)$.

This set-up is more general than is needed. All the channels have the same statistics and all the signals have the same power. Therefore, $\Gamma_k = 1$ for all k, and the term $\prod_{k=1}^{M-1}(1 + \Gamma_k z)$ becomes $\prod_{k=1}^{M-1}(1 + z) = (1 + z)^{M-1}$, which is independent of any index, and from the binomial theorem the coefficient $C_j = \binom{M-1}{j}$. Assuming $M \geq N$ implies $A_i(z)$ never equals unity. These simplifications yield

$$A_i(z) = \frac{1 + \sum_{j=1}^{N-i}\binom{M-1}{j} z^j}{(1 + z)^{M-1}} \tag{36}$$

and the cdf of the SINR after optimal combining yields the result of the theorem.

APPENDIX C

The distribution $F_{\mathrm{MMSE}}(z)$ is of the form $F(z) = 1 - R(z)$, with $R(z)$ given by (34). Combining this with the condition for the limiting distribution to be of type 3, (19), the limiting distribution is of type 3 if the following limit is satisfied:

$$\lim_{z\to\infty} \frac{R(z)\frac{d^2}{dz^2}R(z)}{\left(\frac{d}{dz}R(z)\right)^2} \to 1. \tag{37}$$

Similar to previous analysis, consider the terms that dominate the limit and show that their limit goes to unity. First, expanding the expression for $R(z)$ yields

$$R(z) = \frac{e^{-z/\rho}}{(1+z)^{M-1}} \sum_{i=1}^{N} \frac{\left(\frac{z}{\rho}\right)^{i-1}}{(i-1)!} + \frac{e^{-z/\rho}}{(1+z)^{M-1}} \sum_{i=1}^{N}\sum_{j=1}^{N-i} \binom{M-1}{j} \frac{z^{j+i-1}}{\rho^{i-1}(i-1)!}. \tag{38}$$

All the terms in $R(z)$ decay to zero as z tends to infinity monotonically, so the terms that decay to zero the slowest are of interest. Looking at the first term in $R(z)$, of all the terms in the sum, the one that decays the slowest is when z^i has the largest exponent, thus the dominating term in the limit is

$$\frac{e^{-z/\rho} z^{N-1}}{(1+z)^{M-1}\rho^{N-1}(N-1)!}. \tag{39}$$

Analyzing the second term in $R(z)$, the term that decays the slowest in the summation as z tends to infinity is when z^{j+i-1} has the largest exponent. The largest value the index j in the exponent can take on is $N - i$, and substituting this back into the expression yields z^{N-1} which is independent of the indexes i and j. Therefore, the dominating term is given by

$$\frac{e^{-z/\rho} z^{N-1}}{(1+z)^{M-1}} \sum_{i=1}^{N} \binom{M-1}{N-i} \frac{1}{\rho^i (i-1)!}. \tag{40}$$

Combining (39) and (40) gives the dominating terms of $R(z)$ in the limit

$$\frac{e^{-z/\rho} z^{N-1}}{(1+z)^{M-1}} C \tag{41}$$

where the constant C is defined as

$$C = \frac{1}{\rho^{N-1}(N-1)!} + \sum_{i=1}^{N} \binom{M-1}{N-i} \frac{1}{\rho^i (i-1)!}. \tag{42}$$

From (37), the limiting terms of the first and second derivatives of $R(z)$ are needed, and luckily the dominating term of $R(z)$ in (41) is readily differentiable. Performing some calculus and ignoring terms that decay too fast, the limiting term of $(d/dz)R(z)$ is

$$-C\frac{e^{-z/\rho} z^{N-1}}{\rho(1+z)^{N-1}}, \tag{43}$$

and the limiting term of $(d^2/dz^2)R(z)$ is

$$C\frac{e^{-z/\rho} z^{N-1}}{\rho^2(1+z)^{M-1}}. \tag{44}$$

Combining (41), (43), and (44) yields

$$\lim_{z\to\infty} \frac{R(z)\frac{d^2}{dz^2}R(z)}{\left(\frac{d}{dz}R(z)\right)^2} = \frac{C\frac{e^{-z/\rho}z^{N-1}}{(1+z)^{M-1}} \cdot C\frac{e^{-z/\rho}z^{N-1}}{\rho^2(1+z)^{M-1}}}{\left(-C\frac{e^{-z/\rho}z^{N-1}}{\rho(1+z)^{M-1}}\right)^2} = 1. \tag{45}$$

Therefore the limiting distribution of the maximal order statistic for the SINR after optimal combining is of type 3.

If the scaling rate of the unique solution to $1 - F_{\mathrm{MMSE}}(u_n) = 1/n$ can be found, then the asymptotic scaling rate is known by Equation (15) of [10]

$$\lim_{n\to\infty} \frac{R}{M \log u_n} = 1. \tag{46}$$

For sufficiently large n, $1 - F_{\mathrm{MMSE}}$ is dominated by (41). The solution u_n to $1 - F_{\mathrm{MMSE}}(u_n) = 1/n$ is guaranteed to exist since (41) is monotonically decreasing and continuous. Following the analysis of (21) in [1],

$$\begin{aligned} 1 - F_{\mathrm{MMSE}}(u_n) &= \frac{e^{-u_n/\rho} u_n^{N-1}}{(1+u_n)^{M-1}} C = \frac{1}{n} \\ &\Rightarrow \frac{u_n}{\rho} + (M-1)\log(1+u_n) \\ &\quad - (N-1)\log(u_n) \\ &= \log n - \log C. \end{aligned}$$

For fixed N, M and ρ and sufficiently large n, this yields

$$u_n = \rho \log n + \rho(M-N)\log\log n + O(\log\log\log n)$$

since $\log C$ becomes inconsequential, u_n is monotonically increasing, and $\lim_{n\to\infty}(\log(1+u_n) - \log u_n) = 0$. Thus, the desired scaling rate is achieved.

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